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ABSTRACT

This student text presents instructional materials for a unit of mathematics within the Biomedical Interdisciplinary Curriculum Project (BICP), a two-year interdisciplinary precollege curriculum aimed at preparing high school students for entry into college and vocational programs leading to a career in the health field. Lessons concentrate on biomedical applications in the mathematical area of quadratics, as well as sections on scientific notation and molarity, pH scales, equilibrium and solubility, dissociation constants, chemical equilibria, and operations with complex numbers. (CS)

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BIOMEDICAL MATHEMATICS

UNIT III

QUADRATICS

STUDENT TEXT

REVISED VERSION, 1976

THE BIOMEDICAL INTERDISCIPLINARY CURRICULUM PROJECT

SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION

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SECTION 1: QUADRATIC FUNCTIONS

1-1 Introduction

By now you have seen many examples of the use of functions in studying biomedical problems. Some types of functions are so useful that they arise in not just one, but many different kinds of problems. For example, linear functions were of help in the study of vital capacity and also in predicting the concentration of alcohol in the bloodstream.

In the equation of a linear function, one or both variables appear raised to the first power but never to a higher power. The following are examples of linear functions.

$$y = 6x - 4 \qquad y = 8 \qquad x = -15$$

In this section we will begin the study of a new type of function which appears in many scientific problems. This is the quadratic function which has the general formula

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$. Quadratic functions differ from linear functions in that one of the variables in a quadratic function appears to only the first power but the other variable appears to the second power. In the general formula above y appears to the first power and x appears to the second power. In the general formula above y appears to the first power and x appears to the second power. ax^2 is called the quadratic term, bx the linear term and c the constant term. In order to write the equation of a quadratic function, all we need to know are the values of a, b and c. For example, if a = -3, b = 0 and c = -5, then the quadratic function must be

$$y = -3x^2 - 5$$

EXAMPLE:

A researcher claims that the approximate age of a human embryo or fetus is given by

$$a = 12.8\ m^2 + 9m + 2.3$$

where a is age in months and m is length in meters. Estimate the age of a fetus with a length of .3 meters.

SOLUTION:

This is a quadratic function because one variable (a) appears to the first power and the other (m) appears to the second power. In order to estimate the age we simply substitute .3 for m.

$$\begin{aligned} a &= 12.8(.3)^2 + 9(.3) + 2.3 \\ &= 12.8(.09) + 9(.3) + 2.3 \\ &= 1.152 + 2.7 + 2.3 \\ &\approx 6.2 \text{ months} \end{aligned}$$

When a quadratic function is expressed in the form $y = ax^2 + bx + c$, we say that it is in standard form. Many times we encounter quadratic functions which are not in standard form and we are faced with the problem of converting them into that form.

EXAMPLE:

Put the function $y - x(3 - x) = 0$ into standard form.

SOLUTION:

First we isolate y on the left side of the equation; then we use the distributive law to remove the parentheses.

$$y = x(3 - x)$$
$$y = 3x - x^2$$
$$y = -x^2 + 3x$$

EXAMPLE:

Put the function $3s = (t - 3)^2 - 10$ into standard form.

SOLUTION:

We use binomial multiplication to remove the parentheses.

$$3s = (t - 3)^2 - 10$$
$$3s = (t - 3)(t - 3) - 10$$
$$3s = (t^2 - 6t + 9) - 10$$
$$3s = t^2 - 6t - 1$$

Therefore
$$s = \frac{1}{3}t^2 - 2t - \frac{1}{3}$$

Quadratic functions have many applications. In a couple of weeks you will see how quadratic functions are used in studying the chemistry of digestion. You will examine what happens when antacids and certain food additives reach the stomach. In the meantime we will look at quadratic functions in general with special attention to graphing.

1-2 The graph of $y = x^2$

Recall that the graph of a linear function $y = mx + b$ turns out to be a straight line. What will the graph of a quadratic function look like? This question will be considered in the next few lessons. We begin by looking at the graph of the following simple quadratic function.

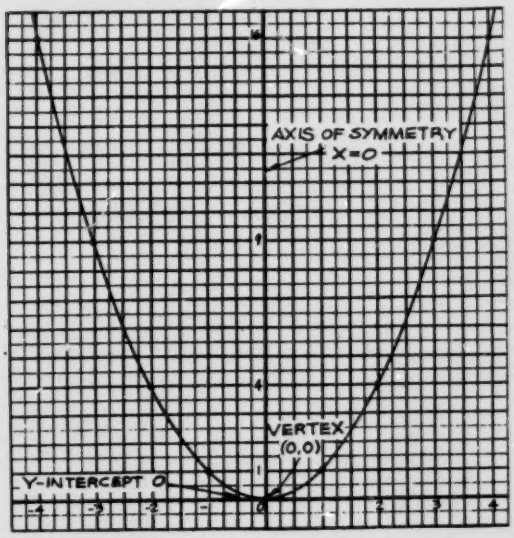
$$y = x^2$$

The first step is to construct a table of values for the function. We compute y-values for integral x-values between -4 and 4, the result is

x	-4	-3	-2	-1	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

When these points are plotted and connected by a smooth curve the result is the graph on the next page. Notice that the two axes are scaled differently. This allows a larger portion of the graph to be displayed in the available space. We will do this frequently in graphing quadratic functions.

Let us note a few important features of the graph. The curve shown is called a parabola. As we graph other quadratic functions you will see that the graphs all look similar. They are also parabolas. The path followed by a pop fly in baseball is a familiar example of a parabola. The profile of half an egg also has a similar shape.



The parabola shown gets wider as we travel upward on the y-axis. We say that the parabola opens upward. The y-intercept for a parabola is defined just as in the case of a linear function. It is the y-coordinate of the point where the graph crosses the y-axis. In the case of $y = x^2$ the graph crosses the y-axis (0, 0). Therefore the y-intercept is 0. Notice that the y-axis splits the graph into two similar pieces. The graph on the left side of the y-axis is a "reflection" of the graph on the right side. We say that the graph is symmetric about the y-axis. The y-axis, or what is the same, the line $x = 0$, is called the axis of symmetry of the graph. The point where the axis of symmetry intersects the parabola is called the vertex. In the case of $y = x^2$ the vertex is the point (0, 0).

PROBLEM SET 1:

1. In each case identify the function as either a linear function or a quadratic function.

a. $y = 3x^2 - x$ c. $3s = 2t + 29$

b. $z = 100$ d. $q^2 + p = 50$

2. The standard form for a quadratic function is $y = ax^2 + bx + c$. In each case write down the equation for the quadratic function with the given values of a, b, and c.

a. $a = 12$ $b = -2$, $c = 3$ c. $a = 2$, $b = 5$, $c = 0$

b. $a = -1$, $b = 0$, $c = 2$ d. $a = 10$, $b = 0$, $c = 0$

3. On Page 1 we introduced the quadratic function

$$a = 12.8\ m^2 + 9m + 2.3$$

where a is the age of a human embryo or fetus in months and m is the length of the fetus in meters. Use this formula to find the approximate age of a fetus which is .2 m long. Round your answer to the nearest .1 month.

Put each of the following quadratic functions into standard form.

4. $y + x^2 - 8 = 0$

5. $3t^2 = s - t$

6. $x(x + 2) - y = 0$

7. $v = (u - 1)(u + 1)$

8. $4y = (x - 4)^2$

9. $s = 2(t + 1)^2 - 1$

10. $z = 4(w - 3)(w - 2)$

11. $y = \frac{1}{2}(x - 2)^2 + 5$

12. $s = (t + 1)^2 + (t - 1)^2$

In the following problems refer to the graph of $y = x^2$ on Page 3 .

13. The equation $y = x^2$ is already in the standard form $y = ax^2 + bx + c$.

    a. a = ? c. c = ?

    b. b = ?

14. The graph is called a _____.

15. The curve opens _____(upward, downward).

16. The line x = 0 is called the _____.

17. The y-intercept of the curve is _____.

18. The point (0, 0) is called the _____ of the curve.

SECTION 2: GRAPHING FUNCTIONS OF THE FORM $y = ax^2 + c$

2-1 How the Value of "a" Affects the Graph

Recall that the graph of a linear function $y = mx + b$ is a straight line. The values of both m and b have an effect on the graph. The quantity m, called the slope, determines how "steep" the line is. The quantity b, called the y-intercept, tells us where the line crosses the y-axis.

It is logical, therefore, to expect that in the case of a quadratic function $y = ax^2 + bx + c$, all three quantities a, b and c will contribute to the shape or location of the graph. How can we best proceed to discover the effect of each one? The answer is that we will use a favorite technique of mathematicians; we will vary the quantities a, b and c one at a time and see how the graph is affected.

In order to study the effect of the quantity a, suppose we let b = 0 and c = 0. Then the general quadratic function $y = ax^2 + bx + c$ reduces to the following

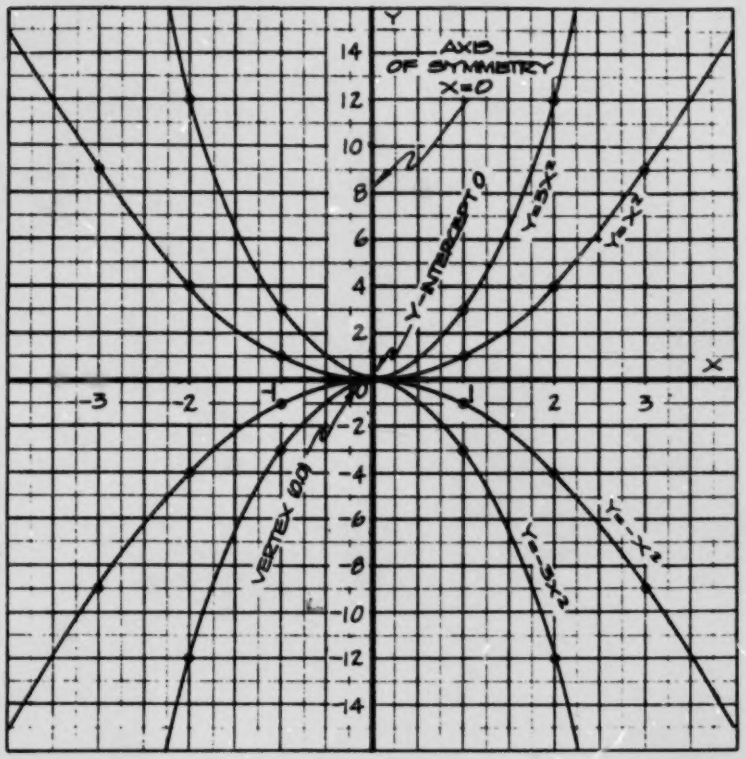
$$y = ax^2$$

In the last section you saw the graph of one function of this type, namely $y = x^2$. For that function, "a" has the value of 1.

How will the graph change if "a" takes on other values? Suppose we let "a" take on the values 3, -1 and -3. The following is a table of values for these quadratic functions.

x	-3	-2	-1	0	1	2	3
$y = 3x^2$	27	12	3	0	3	12	27
$y = -x^2$	-9	-4	-1	0	-1	-4	-9
$y = -3x^2$	-27	-12	-3	0	-3	-12	-27

The graphs of the three functions are shown to the right. The graph of $y = x^2$ is also included for comparison.



How do these graphs differ? They all have the same y-intercept, vertex and axis of symmetry. But two of the parabolas open upward and two downward. The parabolas open upward when "a" is positive (1 and 3) and downward when "a" is negative (-1 and -3).

The parabolas also differ in shape. The graphs of $y = x^2$ and $y = -x^2$ are "flatter" or less steep than those of $y = 3x^2$ and $y = -3x^2$. On the other hand, the graphs of $y = x^2$ and $y = -x^2$ have the same size and shape. If the graphs were made of rigid wire then one of them could be picked up and placed exactly on the other. We say that the graphs of $y = x^2$ and $y = -x^2$ are <u>congruent</u>. Note that the graphs of $y = 3x^2$ and $y = -3x^2$ are also congruent.

The following is a summary of how the value "a" affects the graph of $y = ax^2$.

1. If $a > 0$, the parabola opens upward.

   If $a < 0$, the parabola opens downward.

2. The closer "a" is to zero, the flatter the parabola.

3. If two values of "a" differ only in sign, the corresponding graphs are congruent (same size and shape).

2-2 How the Value of c Affects the Graph

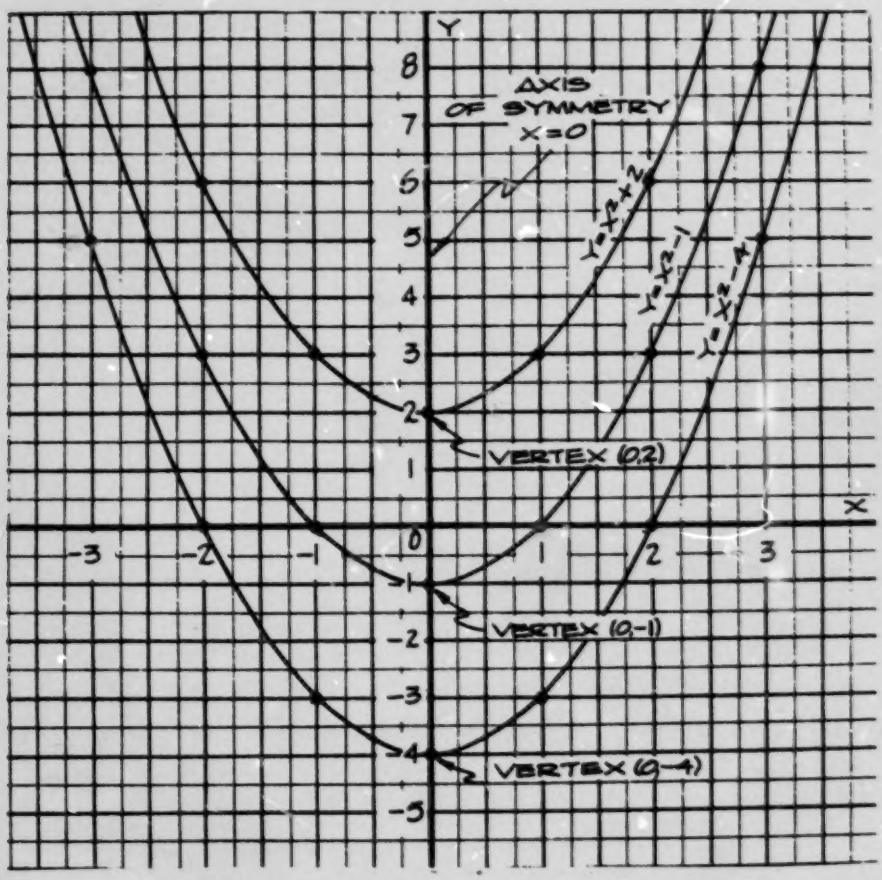
The next step is to pin down how the value of c affects the graph of a quadratic function $y = ax^2 + c$. Accordingly, let us fix $a = 1$ so that our quadratic function takes the form

$$y = x^2 + c$$

The graph on the next page shows what happens when c takes the values -4, -1 and 2.

How do these graphs differ and how are they the same? They are all congruent and have axes of symmetry $x = 0$. But the vertices and y-intercepts differ. The following table shows the pattern involved.

function	vertex	y-intercept
$y = x^2 + 2$	(0, 2)	2
$y = x^2 - 1$	(0, -1)	-1
$y = x^2 - 4$	(0, -4)	-4



On the basis of the table above we might guess that the y-intercept of the function $y = ax^2 + c$ is just c. In fact this can be shown by substituting 0 for x in the equation, $y = ax^2 + c$.

$$y = (a \cdot 0^2) + c$$

$$= c$$

The following points summarize the effect of c on the graph of $y = ax^2 + c$.

1. The graph of $y = ax^2 + c$ has vertex (0, c).
2. The graph of $y = ax^2 + c$ has y-intercept c. The effect of c is to "slide" the graph up or down in the y direction.

PROBLEM SET 2:

In each case make a table of x- and y-values and draw both graphs on the same set of axes.

1. $y = x^2 + 4$
   $y = x^2 - 2$
2. $y = x^2 - \frac{1}{2}$
   $y = -x^2 - \frac{1}{2}$
3. $y = -2x^2 + 8$
   $y = -2x^2 + 2$
4. $y = x^2 + 2$
   $y = \frac{1}{2}x^2 + 2$
5. $y = \frac{1}{2}x^2 + 1$
   $y = -\frac{1}{2}x^2 + 1$

In each case give the y-intercept and vertex of the graph of the function. You need <u>not</u> draw the graph.

6. $y = x^2$
7. $y = x^2 + 1$
8. $y = 3x^2 - 5$
9. $y = \frac{1}{3}x^2 - \frac{1}{6}$
10. $y = 4x^2 + 55$
11. $y = 100x^2 + 81$
12. $y = 30x^2 - \frac{1}{10}$

The following problems require no computation.

13. Does the graph of $y = -4x^2 + 6$ open upward or downward?
14. Which of the following functions has the same vertex as $y = 3x^2 - 2$?
    a. $y = 3x^2 + 1$
    b. $y = -5x^2 - 2$
15. Which function has a graph congruent to that of $y = -4x^2 + 1$?
    a. $y = 4x^2 + 21$
    b. $y = 6x^2 + 1$
16. The graph of a certain quadratic function opens downward, has vertex (0, 7) and is congruent to the graph of $y = 8x^2 + 3$. What is the equation of the function?

SECTION 3: GRAPHING FUNCTIONS OF THE FORM $y = a(x - k)^2 + p$

3-1 How The Value of "k" Affects the Graph

In the last section you saw several graphs of quadratic functions of the form $y = ax^2 + c$. It turned out that we could describe how the values of "a" and "c" influence the graph. The value of "a" determines the shape of the graph and also tells us whether the parabola opens upward or downward. The value of "c" tells us how far above or below the x-axis the vertex will be found.

In this section you will be studying quadratic functions of a different type, those with an equation of the form

$$y = a(x - k)^2 + p$$

where a, k and p are constants and $a \neq 0$. We will try to see how each of the values a, k and p influences the graph. As you can see, the equation above is not in the standard form for a quadratic. We use the above form because it is especially convenient for graphing. You will see why as we go along.

Let us begin by investigating k. We will hold "a" and "p" fixed and vary k. following three functions are examples of what we might get from this procedure.

$$y = (x - 3)^2$$

$$y = (x - 1)^2$$

$$y = (x + 4)^2$$

Notice that a = 1 and p = 0 for all three functions. The only thing that varies is k, taking on values of 3, 1 and -4. You may be tempted to assume that k = 4 in the function $y = (x + 4)^2$, but remember that the expression in parentheses represents x <u>minus</u> k. Therefore

$$x - k = x + 4$$

$$-k = 4$$

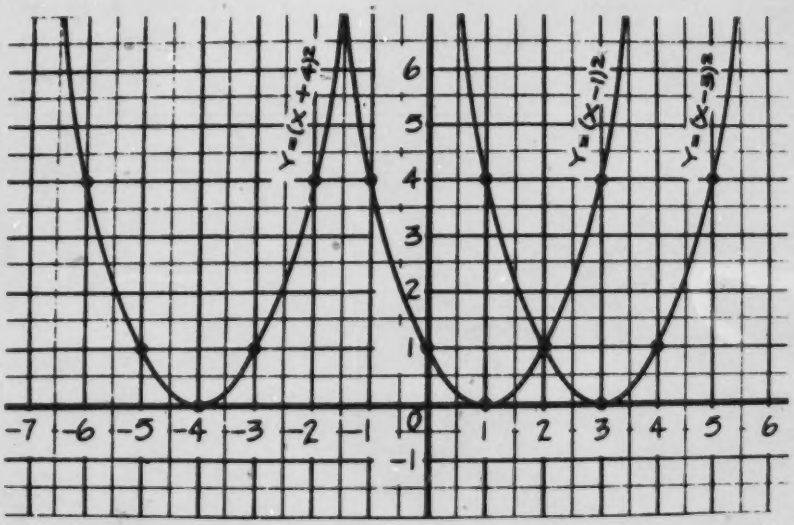
$$k = -4$$

The following shows the three quadratic functions as tables and graphs.

x	$(x + 4)^2$
-6	4
-5	1
-4	0
-3	1
-2	4

x	$(x - 1)^2$
-1	4
0	1
1	0
2	1
3	4

x	$(x - 3)^2$
1	4
2	1
3	0
4	1
5	4



How do these graphs differ? They are all the same size and shape (congruent) and all the vertices are on the x-axis. But the vertices are located at different places on the x-axis. The following table helps show what is going on.

function	k	vertex
$(x - 3)^2$	3	(3, 0)
$(x - 1)^2$	1	(1, 0)
$(x + 4)^2$	-4	(-4, 0)

Notice that k is the same as the x-coordinate of the vertex.

The value of k is the same as the x-coordinate of the vertex. The axis of symmetry is $x = k$.

3-2 How the Value of p Affects the Graph

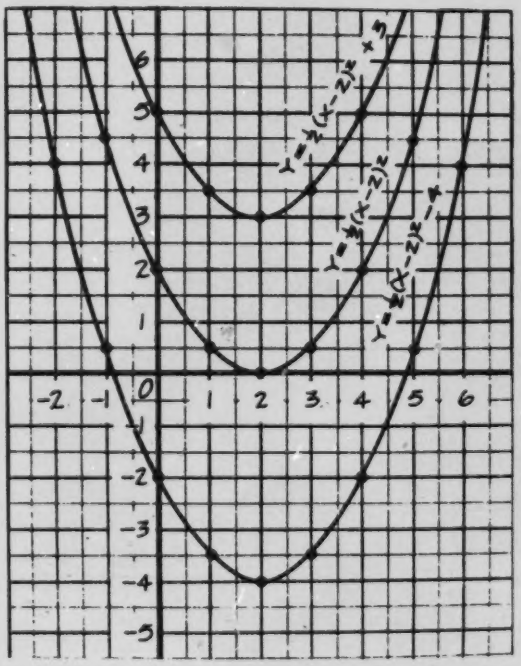
To determine the effect of the quantity p on the graph of a quadratic function $y = a(x - k)^2 + p$, we will fix "a" and k and vary p. For example, let $a = \frac{1}{2}$ and $k = 2$ and let p take on the values of 3, 0 and -4. We then get the following function.

$$y = \tfrac{1}{2}(x - 2)^2 + 3$$

$$y = \tfrac{1}{2}(x - 2)^2 + 0$$

$$y = \tfrac{1}{2}(x - 2)^2 - 4$$

The graphs of these functions are shown to the right.



These graphs are all congruent and the x-coordinate of each vertex is 2. But the y-coordinates of the vertices differ. The table below shows that p is the same as the y-coordinate of the vertex.

function	p	vertex
$\frac{1}{2}(x - 2)^2 + 3$	3	(2, 3)
$\frac{1}{2}(x - 2)^2 + 0$	0	(2, 0)
$\frac{1}{2}(x - 2)^2 - 4$	-4	(2, -4)

In summary, the value of p is the same as the y-coordinate of the vertex.

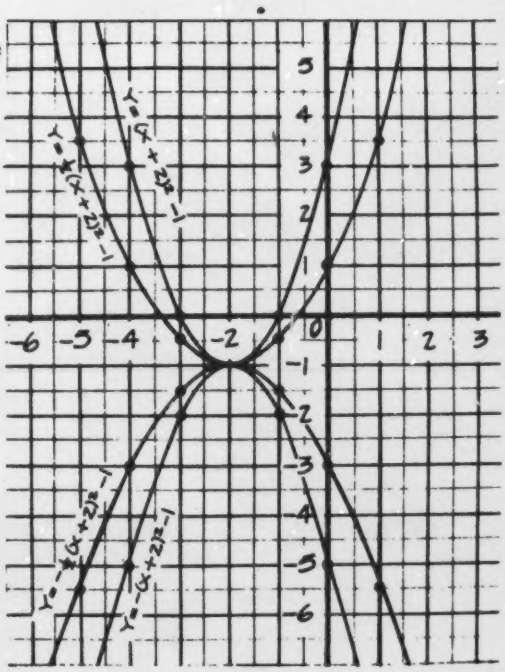
3-3 How the Value of "a" Affects the Graph

In order to discover the influence of the value of "a," we will look at the following functions.

$y = (x + 2)^2 - 1$ $\qquad$ $y = -\frac{1}{2}(x + 2)^2 - 1$

$y = \frac{1}{2}(x + 2)^2 - 1$ $\qquad$ $y = -(x + 2)^2 - 1$

You should convince yourself that k = -2 and p = -1 for all these functions. On the other hand, "a" varies over the values 1, $\frac{1}{2}$, $-\frac{1}{2}$ and -1. The graphs of the four functions are shown to the right.



All these parabolas have the same vertex and axis of symmetry, but the shapes differ. Also two open upward and two downward. In fact the value of "a" has the same influence as in the case of functions of the form $y = ax^2 + c$, which were discussed in Section 2.

1. If a > 0, the parabola opens upward.

   If a < 0, the parabola opens downward.

2. The closer "a" is to zero, the flatter the parabola.

3. If two values of "a" differ only in sign, the corresponding graphs are congruent.

The knowledge gained so far allows us to answer questions about graphs without having to draw them.

EXAMPLE:

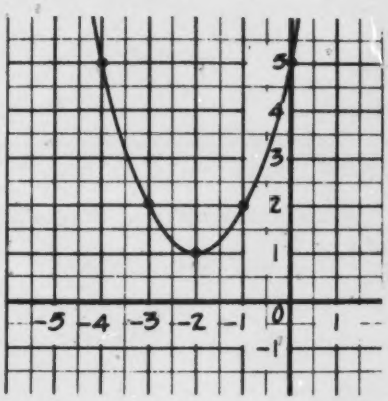
Suppose $y = -5(x + 1)^2 - 10$. Will the graph of this function open upward or downward? Where will the vertex and axis of symmetry be?

SOLUTION:

In this case, a = -5, k = -1 and p = -10. Since "a" is negative, the parabola will open downward. Since k is the x-coordinate of the vertex and p is the y-coordinate of the vertex, the vertex will be at (-1, -10). The axis of symmetry is x = -1.

EXAMPLE:

On the right we have graphed a function of the form $y = (x - k)^2 + p$. Find k and p and write the equation for the function.



SOLUTION:

From the graph it is clear that the vertex is at (-2, 1).

Since k is always the x-coordinate of the vertex, k = -2. Similarly, p is always the y-coordinate of the vertex; so p = 1. Therefore, the equation of the desired function is

$$y = (x + 2)^2 + 1$$

For future reference we summarize the effects of the constants a, k and p on the graph of $y = a(x - k)^2 + p$.

1. If $a > 0$, the parabola opens upward.

   If $a < 0$, the parabola opens downward.

2. The closer "a" is to zero, the flatter the parabola.

3. If two values of "a" differ only in sign, the corresponding graphs are congruent.

4. The axis of symmetry is the line $x = k$.

5. The vertex is at the point (k, p).

PROBLEM SET 3:

In the graph of $y = a(x - k)^2 + p$,

1. the x-coordinate of the vertex is _____(a, k, p).

2. the y-coordinate of the vertex is _____(a, k, p).

3. To decide whether the parabola opens upward or downward we have to check the value of _____(a, k, p).

4. The axis of symmetry of the graph is the line $x =$ _____(a, k, p).

Find the values of a, k and p, the vertex and the axis of symmetry of each of the following equations. You need not draw the graph.

5. $y = (x - 1)^2 + 1$

6. $y = -\frac{1}{2}(x - 3)^2 + 2$

7. $y = -2(x - 8)^2 + 5$

8. $y = 6(x + 5)^2 - 1$

9. $y = -(x + 7)^2 - 4$

10. $y = -2(x + 10)^2 + 5$

Plot the graphs of each of the following pairs of equations on the same set of coordinate axes.

11. $y = (x + 1)^2$
    $y = (x - 1)^2$

12. $y = (x - 2)^2 + 1$
    $y = -(x - 2)^2 + 1$

13. $y = (x - 3)^2$
    $y = -(x + 3)^2$

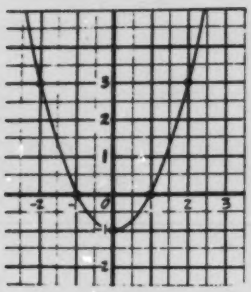
14. $y = 2(x + 1)^2 - 1$
    $y = (x + 1)^2 - 1$

15. $y = -(x - 2)^2$
    $y = -(x - 2)^2 - 2$

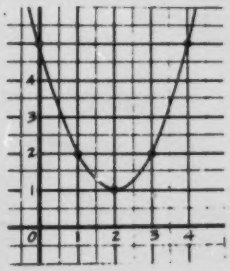
16. $y = 2(x - 1)^2 + 1$
    $y = (x - 1)^2 - 1$

Each of the following is the graph of a function of the form $y = (x - k)^2 + p$. Find the values of k and p by inspecting the graph and write the equation for the function.

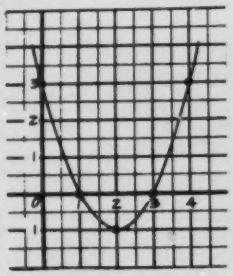
17.



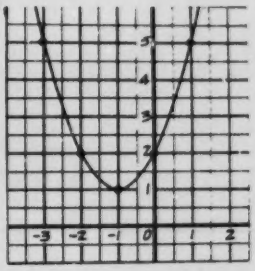
18.



19.



20.



21. A certain quadratic function has a graph which opens downward and which is congruent to the graph of $y = 2x^2$. The vertex is at the point (4, -3). Write the equation of the function.

22. A certain quadratic function has a graph which opens upward and which is congruent to the graph of $y = 3(x - 1)^2 + 2$. The axis of symmetry is x = 10 and the y-coordinate of the vertex is 7. Write the equation of the function in <u>standard form</u>.

SECTION 4: COMPLETING THE SQUARE (PART I)

4-1 <u>Putting Functions in the Form $y = (x - k)^2$</u>

Suppose that you are studying the quadratic function

$$y = 3x^2 - 12x + 1$$

and you want to know where the vertex of the graph will be. This is a difficult problem as it stands, but it is easy if we know that the function can be written another way.

$$y = 3(x - 2)^2 - 11$$

The function is now in the form

$$y = a(x - k)^2 + p$$

The last section showed that the vertex will be at the point (k, p), that is, at the point (2, -11).

The above example illustrates the fact that for purposes of graphing, it is convenient to express a quadratic function in the form

$$y = a(x - k)^2 + p$$

With this in mind, we are going to develop in the next two sections a method called <u>completing the square</u>, for converting any quadratic function into this form.

In this section, we will examine functions such that the <u>coefficient of x^2 is one</u> (a = 1). First we ask, "When can such a function be written as a perfect square $y = (x - k)^2$?" On the following page we will look at an example with k = 6.

$$y = (x - 6)^2$$
$$= x^2 - (2 \cdot 6)x + 6^2$$
$$= x^2 - 12x + 36$$

The function is now in standard form $y = ax^2 + bx + c$. How are the coefficients of this expression related? There are several things to notice.

1. The constant term, c, is equal to the square of k. That is, $36 = 6^2$.

2. The coefficient, b, of x is equal to -2k. That is, $-12 = -2 \cdot 6$.

3. The constant term, c, is equal to the square of half the coefficient, b, of x. That is, $36 = \left(\frac{-12}{2}\right)^2$.

It turns out that these observations hold for any function of the form $y = (x - k)^2$. We can use Property #3 to decide whether a given quadratic function is expressible in the form $y = (x - k)^2$.

EXAMPLE 1:

Decide whether the function $y = x^2 - 20x + 100$ can be written in the form $y = (x - k)^2$, and if so, find the value of k.

SOLUTION:

First, we check to see whether the function satisfies Property #3 above. The coefficient of x is -20 and the constant term is 100. The square of half the coefficient of x is

$$\left(\frac{-20}{2}\right)^2 = 100$$

Since this agrees with the constant term, the function can be written in the form $y = (x - k)^2$.

Second, we find the value of k by using Property #2, which says that the coefficient of x is equal to -2k. Therefore

$$-2k = -20$$
$$k = 10$$

Therefore, the given function can be written as

$$y = (x - k)^2$$
$$= (x - 10)^2$$

EXAMPLE 2:

Can the function $y = x^2 + \frac{1}{3}x + \frac{1}{36}$ be written as a perfect square $y = (x - k)^2$? If so, what is the value of k?

SOLUTION:

First we compute the square of half the coefficient of x.

$$\left(\frac{1}{2} \cdot \frac{1}{3}\right)^2 = \left(\frac{1}{6}\right)^2$$
$$= \frac{1}{36}$$

This is the same as the constant term, and so the function is a perfect square.

Second, we use Property #2 to find k.

$$-2k = \frac{1}{3}$$

$$k = -\frac{1}{6}$$

Therefore,

$$y = (x - k)^2$$
$$= [x - (-\tfrac{1}{6})]^2$$
$$= \left(x + \frac{1}{6}\right)^2$$

EXAMPLE 3:

Can the function $y = x^2 - 20x$ be written as a perfect square $y = (x - k)^2$?

SOLUTION:

The square of half the coefficient of x is

$$\left(\frac{-20}{2}\right)^2 = 100$$

This certainty doesn't agree with the constant term, which is zero in this case. Therefore, $x^2 - 20x$ is not a perfect square.

4-2 Putting Functions in the Form $y = (x - k)^2 + p$

The function $y = x^2 - 20x$ of the last example cannot be put in the form $y = (x - k)^2$. However, it can be put in a closely related form. To accomplish this, we use the information in Example 1 on Page 13. In that example, we found that

$$y = x^2 - 20x + 100$$
$$= (x - 10)^2$$

Notice that the functions $y = x^2 - 20x$ and $y = x^2 - 20x + 100$ differ by 100. We can use this fact to write

$$y = x^2 - 20x$$
$$= (x^2 - 20x + 100) - 100$$
$$= (x - 10)^2 - 100$$

The function $y = x^2 - 20x$ is now in the form

$$y = (x - k)^2 + p$$

with $k = 10$ and $p = -100$.

In this example we added 100 to the function $y = x^2 - 20x$ and subtracted it at the same time. Since $+100 - 100 = 0$, the function remains unchanged. You may see why this technique is called completing the square. We started with an expression that was not a square and "completed" it by adding the necessary constant, and then subtracted the same constant. The constant was the square of half the coefficient of x. This method can always be used.

EXAMPLE 4:

Is the function $y = x^2 + 10x + 7$ a perfect square? If not, put it in the form $y = (x - k)^2 + p$.

SOLUTION:

First we compute the square of half the coefficient of x.

$$\left(\frac{10}{2}\right)^2 = 25$$

This is not equal to the constant term; the function is not a perfect square.

Second, we continue by adding and subtracting 25.

$$y = x^2 + 10x + 7$$
$$= x^2 + 10x + 7 + 25 - 25$$

Rearranging, we get

$$y = (x^2 + 10x + 25) + 7 - 25$$

The quantity in parentheses is the square $(x + 5)^2$. Therefore

$$y = (x + 5)^2 + 7 - 25$$
$$= (x + 5)^2 - 18$$

The function is now in the form

$$y = (x - k)^2 + p$$

with $k = -5$ and $p = -18$.

PROBLEM SET 4:

Problems 1 through 7 refer to the following equations with $k = 7$.

$$(x - k)^2 = (x - 7)^2$$
$$= x^2 - (2 \cdot 7)x + 7^2$$
$$= x^2 - 14x + 49$$

1. $k^2 = ?$
2. (Multiple Choice) k^2 has the same value as
   a. the coefficient of x^2.
   b. the coefficient of x.
   c. the constant term.
3. What is the coefficient of x?
4. (Multiple Choice) The coefficient of x is equal to
   a. 2k
   b. -2k
   c. k^2
5. What is half the coefficient of x(numerical value)?
6. What is the square of half the coefficient of x(numerical value)?
7. (Multiple Choice) The square of half the coefficient of x is equal to
   a. the coefficient of x^2.
   b. the coefficient of x.
   c. the constant term.

8. a. Write the equation $y = (x - k)^2$ with -9 substituted for k.

   b. Put the function of Part a in standard form.

   c. What is the square of half the coefficient of x for this function?

   d. What is the constant term?

   e. Do the answers to Parts c and d agree?

In each of the following problems decide whether the function is a perfect square $y = (x - k)^2$. If so, write the function as a square with the proper value of k substituted.

9. $y = x^2 - 8x + 16$
10. $y = x^2 + 16x + 64$
11. $y = x^2 - 6x + 10$
12. $y = x^2 + 50x + 625$
13. $y = x^2 + 18x + 224$
14. $y = x^2 - 14x - 49$
15. $y = x^2 + \frac{1}{8}x + \frac{1}{16}$
16. $y = x^2 - 3x + \frac{9}{2}$

In each of the following problems, find the number which must be added to the function in order to obtain a perfect square.

17. $y = x^2 - 6x$
18. $y = x^2 + 12x$
19. $y = x^2 - 40x$
20. $y = x^2 + 10x$
21. $y = x^2 + \frac{1}{5}x$
22. $y = x^2 - 5x$

In each of the following problems, put the given function in the form $y = (x - k)^2 + p$.

23. $y = x^2 - 10x + 3$
24. $y = x^2 + 22x - 10$
25. $y = x^2 + 16x$
26. $y = x^2 - 30x + 230$
27. $y = x^2 - 3x$
28. $y = x^2 + 7x - 1$
29. $y = x^2 + 11x$
30. $y = x^2 + \frac{5}{2}x + \frac{41}{16}$

SECTION 5: COMPLETING THE SQUARE (PART II)

At the beginning of Section 4, the following function appeared.

$$y = 3x^2 - 12x + 1$$

It was pointed out that this function can also be converted to

$$y = 3(x - 2)^2 - 11$$

We are now ready to discuss how this conversion is accomplished. That is, we will discuss how a function can be put in the form

$$y = a(x - k)^2 + p$$

This problem resembles those of the last section, but there is a new feature. In Section 4, the coefficient, a, of x^2 was always 1. This time the coefficient is not 1, but 3. A slightly different technique is required to solve this problem, but mainly you will make use of what was learned in the last section.

We begin by factoring the coefficient 3 from the first two terms.

$$y = 3x^2 - 12x + 1$$
$$= 3(x^2 - 4x) + 1$$

Next we concentrate on the expression $x^2 - 4x$. The coefficient of x^2 is 1 so we can use the square completing technique of the last section. We add and subtract the square of half the coefficient of x, that is, $\left(\frac{-4}{2}\right)^2 = 4$.

$$y = 3(x^2 - 4x) + 1$$
$$= 3(x^2 - 4x + 4 - 4) + 1$$

The next step is to remove the -4 term from the parentheses. To do so, we must multiply it by 3, obtaining -12.

$$y = 3(x^2 - 4x + 4 - 4) + 1$$
$$= 3(x^2 - 4x + 4) - 12 + 1$$
$$= 3(x^2 - 4x + 4) - 11$$

The quantity in parentheses is now the perfect square $(x - 2)^2$.

$$y = 3(x^2 - 4x + 4) - 11$$
$$= 3(x - 2)^2 - 11$$

The function is now in the desired form $y = a(x - k)^2 + p$.

The same procedure can be used on any quadratic function in the form $y = ax^2 + bx + c$. For convenience we list the steps.

1. Factor the coefficient "a" of x^2 from the first two terms.
2. Find the square of half the coefficient of x.
3. Add and subtract the amount of Step 2 inside the parentheses.
4. Remove the subtracted constant from the parentheses by multiplying it by "a."
5. Write the remaining expression in parentheses as a perfect square and add the constant terms.

The following example will illustrate this procedure step by step.

EXAMPLE:

Put the function

$$y = -4x^2 + 10x - 6$$

in the form

$$y = a(x - k)^2 + p$$

SOLUTION:

Step 1: We factor the coefficient of x^2 from the first two terms.

$$y = -4x^2 + 10x - 6$$
$$= -4(x^2 - \tfrac{5}{2}x) - 6$$

Step 2: The coefficient of x is $-\frac{5}{2}$. The square of half this number is

$$\left[\tfrac{1}{2}(-\tfrac{5}{2})\right]^2 = \frac{25}{16}$$

Step 3: We add and subtract inside the parentheses.

$$y = -4(x^2 - \frac{5}{2}x) - 6$$

$$= -4(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}) - 6$$

Step 4: The $-\frac{25}{16}$ is removed from the parentheses. To do so we multiply it by -4, obtaining $\frac{25}{4}$

$$y = -4(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}) - 6$$

$$= -4(x^2 - \frac{5}{2}x + \frac{25}{16}) + \frac{25}{4} - 6$$

Step 5: We write the expression in parentheses as a square and add the constant terms.

$$y = -4(x^2 - \frac{5}{2}x + \frac{25}{16}) + \frac{25}{4} - 6$$

$$= -4(x - \frac{5}{4})^2 + \frac{25}{4} - \frac{24}{4}$$

$$= -4(x - \frac{5}{4})^2 + \frac{1}{4}$$

The function is now in the form $y = a(x - k)^2 + p$ with $a = -4$, $k = \frac{5}{4}$ and $p = \frac{1}{4}$.

PROBLEM SET 5:

In each of the following problems, factor the coefficient of x^2 from the first two terms. That is, perform Step 1 of the square-completion process.

EXAMPLE:

$5x^2 - 35x + 18 = 5(x^2 - 7x) + 18$

1. $12x^2 + 12x + 1$
2. $-3x^2 - 9x + 11$
3. $2x^2 - 22x + 13$
4. $-4x^2 + 14x - 6$
5. $6x^2 - x$
6. $5x^2 + 2x + 7$
7. $8x^2 + 12x - 100$
8. $-1000x^2 - 100x - 10$

In each of the following problems perform Step 4 of the square-completion process. That is, remove the subtracted constant from the parentheses.

EXAMPLE:

$3(x^2 - 6x + 9 - 9) + 10 = 3(x^2 - 6x + 9) - 27 + 10$

9. $2(x^2 + 2x + 1 - 1) + 6$
10. $-4(x^2 - 8x + 16 - 16) - 11$
11. $5(x^2 - 4x + 4 - 4) + 66$
12. $-3(x^2 + 20x + 100 - 100) + 200$
13. $7(x^2 - 10x + 25 - 25) - 60$
14. $-10(x^2 + 18x + 81 - 81)$

Put each of the following functions in the form $y = a(x - k)^2 + p$.

15. $y = 3x^2 + 6x + 1$
16. $y = -2x^2 + 8x$
17. $y = 4x^2 - 40x + 119$
18. $y = -x^2 + 2x - 2$
19. $y = 5x^2 + 30x + 47$
20. $y = 2x^2 - 24x + 72$
21. $y = 2x^2 - 2x + \frac{1}{2}$
22. $y = -3x^2 - 2x$
23. $y = 4x^2 + 4x + 3$
24. $y = \frac{1}{2}x^2 - 2x$
25. $y = -\frac{1}{3}x^2 + \frac{1}{3}x + \frac{1}{12}$

SECTION 6: MANIPULATION OF RADICAL EXPRESSIONS

6-1 Radicals

In Section 1 we mentioned one quadratic function relating the length of a fetus or embryo to its age in months. Another quadratic relationship which is often used in Haase's Rule which says that

$$c = a^2$$

where c is length in centimeters and a is age in months. This is a quadratic relationship because one of the variables appears to the first power and the other to the second power. The relationship is considered reliable until the end of the fifth month of pregnancy.

According to Haase's Rule, what is the approximate age of a fetus 16 cm long? We substitute 16 for c, obtaining the equation

$$a^2 = 16$$

In this case, a must be a number with a square of 16. That is

$$a^2 = 16$$

$$a = \pm\sqrt{16}$$

$$= \pm 4$$

Although both the values, 4 and -4, satisfy the equation, an age of -4 months makes no sense. Therefore, we are left with the correct answer of 4 months.

In problems involving quadratic functions, you will often encounter numbers under square root signs. Such expressions are called radical expressions, and the signs are radical signs.

6-2 Use of a Square Root Table

Suppose that we want to use Haase's Rule to estimate the age of a fetus with a length of 23 cm. In this case

$$a^2 = 23$$

and

$$a = \pm\sqrt{23}$$

Since 23 is not a perfect square, the approximate value of $\sqrt{23}$ must be found through the use of a calculator or square root table. The value is easily found by referring to the Square Root Table at the end of this book. We find the number 23 in the column labeled "n" and read the approximate square root 4.796 in the column labeled "$\sqrt{n}$." Therefore

$$a \approx \pm 4.796$$

Choosing the positive value and rounding to the nearest tenth of a month we obtain

$$a \approx 4.8 \text{ months}$$

The following examples show further use of the table.

EXAMPLE:

Express $\sqrt{453}$ to the nearest tenth.

SOLUTION:

The number 5150 does not appear in the "n" column. However 515 does appear and $5150 = 10 \cdot 515$. If we let $n = 515$, then $10n = 5150$ and $\sqrt{10n} = \sqrt{5150}$. Therefore we find 515 in the "n" column and read the value 71.76 from the "$\sqrt{10n}$" column. Rounding this value to the nearest tenth we obtain

$$\sqrt{5150} \approx 71.8$$

6-3 Manipulation of Radicals

Suppose that we encounter the problem of computing $\sqrt{91600}$. The number 91600 does not appear in the "n" column. Certainly $91600 = 10 \times 9160$ but 9160 does not appear in the "n" column either, so we cannot use the "10n" column. However, notice that

$$\begin{aligned} 91600 &= 100 \times 916 \\ &= 10^2 \times 916 \end{aligned}$$

When square roots are taken, the perfect square 10^2 can be removed from the radical sign.

$$\begin{aligned} \sqrt{91600} &= \sqrt{10^2 \times 916} \\ &= \sqrt{10^2} \times \sqrt{916} \\ &= 10\sqrt{916} \end{aligned}$$

The property of radical expressions which is used above can be written algebraically as follows.

$$\boxed{\sqrt{s^2 r} = s\sqrt{r}}$$

Returning to the example, we look up $\sqrt{916}$ in the Square Root Table at the end of the book. We find

$$\sqrt{916} \approx 30.27$$

Therefore

$$\begin{aligned} \sqrt{91600} &= 10\sqrt{916} \\ &\approx 10(30.27) \\ &\approx 302.7 \end{aligned}$$

The following example shows the use of another property of radicals.

EXAMPLE:

Find $\sqrt{6.75}$

SOLUTION:

The number 6.75 does not appear in the table, but 675 does. We notice that

$$\begin{aligned} 6.75 &= \frac{675}{100} \\ &= \frac{675}{10^2} \end{aligned}$$

When square roots are taken, the perfect square 10^2 can be removed from the radical sign.

$$\sqrt{6.75} = \sqrt{\frac{675}{10^2}}$$

$$= \frac{\sqrt{675}}{\sqrt{10^2}}$$

$$= \frac{\sqrt{675}}{10}$$

The property of radical expressions which is used above can be written algebraically as follows.

$$\boxed{\sqrt{\frac{r}{s^2}} = \frac{\sqrt{r}}{s}}$$

Returning to the example, we look up $\sqrt{675}$ in the table.

$$\sqrt{675} \approx 25.98$$

Therefore

$$\sqrt{6.75} = \frac{\sqrt{675}}{10}$$

$$\approx \frac{25.98}{10}$$

$$\approx 2.598$$

6-4 A Final Set of Examples

EXAMPLE:

Find the following values

$$\sqrt{4.31}, \quad \sqrt{43.1}, \quad \sqrt{431}, \quad \sqrt{4310}$$

SOLUTION:

$\sqrt{431}$ can be found directly from the table

$$\sqrt{431} \approx 20.76$$

Since $4310 = 10 \times 431$ we can read $\sqrt{4310}$ from the "$\sqrt{10n}$" column.

$$\sqrt{4310} \approx 65.65$$

In order to compute the remaining roots, first notice that

$$4.31 = \frac{431}{100}$$

Therefore

$$\sqrt{4.31} = \sqrt{\frac{431}{10^2}}$$

$$= \frac{\sqrt{431}}{10}$$

$$\approx \frac{20.76}{10}$$

$$\approx 2.076$$

To compute $\sqrt{43.1}$, notice that

$$43.1 = \frac{4310}{100}$$

Therefore

$$\sqrt{43.1} = \sqrt{\frac{4310}{10^2}}$$

$$= \frac{\sqrt{4310}}{10}$$

$$\approx \frac{65.65}{10}$$

$$\approx 6.565$$

PROBLEM SET 6:

Find the approximate value for each of the following square roots. You will need to use the "n" and "$\sqrt{n}$" columns in the Square Root Table at the end of the book.

1. $\sqrt{13}$
2. $\sqrt{40}$
3. $\sqrt{816}$
4. $\sqrt{352}$
5. $\sqrt{555}$

Find the approximate value for each of the following. You will need to use the "n" and "$\sqrt{10n}$" columns.

6. $\sqrt{2110}$
7. $\sqrt{4500}$
8. $\sqrt{7320}$
9. $\sqrt{9980}$
10. $\sqrt{1250}$

In each of the following problems find the unknown number x.

11. $37200 = x \cdot 372$
12. $57100 = x \cdot 571$
13. $3.78 = \frac{378}{x}$
14. $6.93 = \frac{693}{x}$
15. $76.9 = \frac{7690}{x}$

Find the value of y in each of the following problems. Use the property $\sqrt{s^2 r} = s\sqrt{r}$.

16. $\sqrt{4^2 \cdot 3} = y\sqrt{3}$
17. $\sqrt{5^2 \cdot 18} = y\sqrt{18}$
18. $\sqrt{1200} = y\sqrt{12}$
19. $\sqrt{36500} = y\sqrt{365}$
20. $\sqrt{217} = 10\sqrt{y}$

Find the value of y. Use the property $\sqrt{\frac{r}{s^2}} = \frac{\sqrt{r}}{s}$.

21. $\sqrt{\frac{17}{6^2}} = \frac{\sqrt{17}}{y}$
22. $\sqrt{\frac{31}{7^2}} = \frac{\sqrt{y}}{7}$
23. $\sqrt{\frac{119}{25}} = \frac{\sqrt{119}}{y}$
24. $\sqrt{3.34} = \frac{\sqrt{334}}{y}$
25. $\sqrt{70.7} = \frac{\sqrt{7070}}{y}$

In each of the following problems the length of a human fetus is given. Use Haase's Rule to estimate its age to the nearest tenth of a month.

36. 20 cm 37. 11 cm 38. 7 cm

SECTION 7: QUADRATIC EQUATIONS

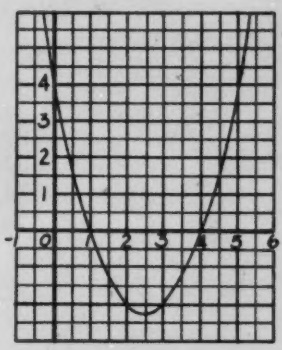
7-1 The Number of Roots

In the coming sequence on applied quadratics you will study problems which lead to quadratic equations of the form

$$ax^2 + bx + c = 0$$

where a, b and c are specific numbers. We say that a quadratic equation is solved when we have found all the values of x, called roots, which make the equation true. In the last section it was pointed out that quadratic equations of the form $x^2 - c = 0$ (or $x^2 = c$) have the roots $\pm\sqrt{c}$.

In order to explore the relationship between quadratic functions and quadratic equations suppose we are given a quadratic function, for example, $y = x^2 - 5x + 4$. This function is shown on the first graph to the right.

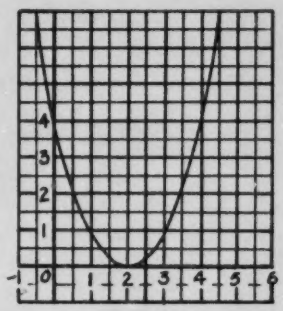


$y = x^2 - 5x + 4$

Which values of x make this quadratic function $y = x^2 - 5x + 4$ equal to zero? From the graph we see that $y = 0$ when $x = 1$ and $x = 4$. Therefore 1 and 4 are the roots of the quadratic equation

$$x^2 - 5x + 4 = 0$$

We have just looked at a quadratic equation which has two real roots. However, not all quadratic equations have two real roots. The examples are shown in the second and third graphs to the right.



$y = x^2 - 4x + 4$

The parabola shown on the second graph to the right touches the x-axis at only one point. There is only one value of x, namely 2, which makes the expression $x^2 - 4x + 4$ equal to zero. Therefore the quadratic equation

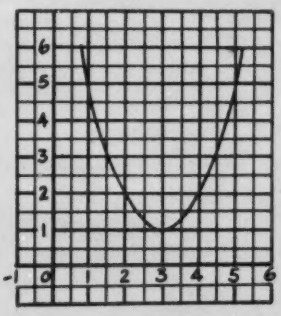
$$x^2 - 4x + 4 = 0$$

has only one real root.

The third graph to the right shows another possibility. In this case there are no x-intercepts at all. There are no real values of x which make the expression $x^2 - 6x + 10$ equal to zero. Therefore the quadratic equation

$$x^2 - 6x + 10 = 0$$

has no real roots.



$y = x^2 - 6x + 10$

The above discussion can be summarized as follows. A quadratic equation will have either one real root, two real roots or no real roots.

7-2 Solving Quadratic Equations by Square Completion

You will probably be glad to learn that there is a way to find the roots of a quadratic equation without drawing the graph of the function. In fact, we can use the square completion technique developed in Section 5.

EXAMPLE 1:

Find the roots of the quadratic equation

$$x^2 - 5x + 6 = 0$$

SOLUTION:

The first part of the solution involves completing the square in the quadratic expression. If you have trouble following the steps, have a look at Section 5 and review the technique.

$$x^2 - 5x + 6 = 0$$

$$(x^2 - 5x) + 6 = 0$$

$$(x^2 - 5x + \frac{25}{4}) + 6 - \frac{25}{4} = 0$$

$$(x - \frac{5}{2})^2 - \frac{1}{4} = 0$$

The square completion is now accomplished. Next the constant term is transferred to the right side of the equation.

$$(x - \frac{5}{2})^2 = \frac{1}{4}$$

Now we take the square root of each side.

$$\sqrt{(x - \frac{5}{2})^2} = \pm\sqrt{\frac{1}{4}}$$

$$x - \frac{5}{2} = \pm\frac{1}{2}$$

Isolating x on the left side we get

$$x = \frac{5}{2} \pm \frac{1}{2}$$

This expression represents the two roots of the equation. They are

$$\frac{5}{2} + \frac{1}{2} = 3 \quad \text{and} \quad \frac{5}{2} - \frac{1}{2} = 2$$

EXAMPLE 2:

Find the roots of the equation

$$2x^2 + 8x + 5 = 0$$

SOLUTION:

We complete the square and take square roots.

$$2x^2 + 8x + 5 = 0$$

$$2(x^2 + 4x) + 5 = 0$$

$$2(x^2 + 4x + 4 - 4) + 5 = 0$$

$$2(x^2 + 4x + 4) + 5 - 8 = 0$$

$$2(x + 2)^2 - 3 = 0$$

$$2(x + 2)^2 = 3$$

$$(x + 2)^2 = 1.5$$

$$x + 2 = \pm\sqrt{1.5}$$

$$x + 2 = \pm 1.225 \text{ (from Square Root Table)}$$

$$x = -2 \pm 1.225$$

So the approximate roots are

$$-2 + 1.225 = -.775 \quad \text{and} \quad -2 - 1.225 = -3.225$$

EXAMPLE 3:

Find the roots of the equation

$$4x^2 - 12x + 9 = 0$$

SOLUTION:

The square completion steps, which have been omitted, lead to the equation

$$4(x - \tfrac{3}{2})^2 = 0$$

or

$$(x - \tfrac{3}{2})^2 = 0$$

Taking square roots of both sides gives

$$x - \tfrac{3}{2} = 0$$

Therefore

$$x = \tfrac{3}{2}$$

and the equation has only <u>one</u> root.

EXAMPLE 4:

Find the roots of the equation

$$x^2 - 4x + 10 = 0$$

SOLUTION:

The square completion steps (omitted) lead to the equation

$$(x - 2)^2 + 6 = 0$$

or

$$(x - 2)^2 = -6$$

Taking square roots of both sides gives

$$x - 2 = \pm\sqrt{-6}$$

This statement is clearly absurd. A negative number does not have a real square root. Therefore the original quadratic equation has no real roots at all.

PROBLEM SET 7:

1. Which of the following is a quadratic equation.

   a. $y = 3x - 2$

   b. $y = x^2 + 6x - 24$

   c. $7x^2 + 100x - 11 = 0$

   d. $5x + 6 = 0$

2. (T or F) Some quadratic equations have no real roots.

3. (T or F) A quadratic equation cannot have just one real root.

4. (T or F) Some quadratic equations have three real roots.

5. (T or F) The only way to solve a quadratic equation is by graphing the corresponding quadratic function.

In each of the following problems find the root or roots of the quadratic equation. If there are no real roots write "NR."

6. $x^2 + 2x - 8 = 0$

7. $x^2 - 6x + 9 = 0$

8. $2x^2 - 12x + 10 = 0$

9. $2x^2 + 12x + 20 = 0$

10. $x^2 - x - 6 = 0$

11. $x^2 - 10x + 25 = 0$

12. $x^2 + 3x + 16 = 0$

13. $x^2 + x - 2 = 0$

14. $3x^2 + 18x + 24 = 0$

15. $x^2 - 7x + 10 = 0$

16. $x^2 + 12x + 36 = 0$

17. $x^2 - 6x + 1 = 0$

18. $x^2 + 2x - 5 = 0$

19. $x^2 - 5x + 3 = 0$

20. $2x^2 + 8x - 5 = 0$

In the following quadratic equations some or all of the coefficients are letters. Solve the equations by square completion. Each answer will be a formula containing the letter or letters.

21. $x^2 + x + c = 0$

22. $x^2 - x - c = 0$

23. $x^2 + bx + 1 = 0$

24. $ax^2 + bx + c = 0$

SECTION 8: THE QUADRATIC FORMULA

8-1 The Quadratic Formula

In the last section you saw how square completion can be used to find the real roots of any quadratic equation. If the square-completion technique is used on the standard quadratic equation

$$ax^2 + bx + c = 0$$

then we obtain the following formula for the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is called the quadratic formula. We will not concern ourselves with deriving it, but your teacher may choose to show you the steps.

In the quadratic formula, the roots of the equation are written in terms of the coefficients, a, b and c. An example will illustrate how the formula is used.

EXAMPLE 1:

In Section 7, Example 1 involved the quadratic equation

$$x^2 - 5x + 6 = 0$$

Find the roots of this equation using the quadratic formula.

SOLUTION:

First we note the values of the coefficients.

$$a = 1 \quad b = -5 \quad c = 6$$

Next we substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-5) \pm \sqrt{(-5)^2 - (4 \cdot 1 \cdot 6)}}{2 \cdot 1}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm 1}{2}$$

Therefore, the two roots are

$$\frac{5 + 1}{2} = 3 \quad \text{and} \quad \frac{5 - 1}{2} = 2$$

8-2 More Examples

The next few examples show the cases of two real roots, one real root and no real roots, as they arise through the quadratic formula.

EXAMPLE 2:

Find the roots of the equation

$$2x^2 + 7x - 1 = 0$$

SOLUTION:

In this case a = 2, b = 7, c = -1. We substitute into the quadratic formula.

$$x = \frac{-7 \pm \sqrt{7^2 - [4 \cdot 2 \cdot (-1)]}}{2 \cdot 2}$$

$$= \frac{-7 \pm \sqrt{57}}{4}$$

Using the Square Root Table, we find $\sqrt{57} \approx 7.550$. So the roots are

$$\frac{-7 + 7.550}{4} \approx 0.138 \quad \text{and} \quad \frac{-7 - 7.550}{4} \approx -3.638$$

EXAMPLE 3:

Find the roots of the equation

$$x^2 - 10x + 25 = 0$$

SOLUTION:

We have a = 1, b = -10 and c = 25. Therefore

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - (4 \cdot 1 \cdot 25)}}{2 \cdot 1}$$

$$= \frac{10 \pm \sqrt{0}}{2}$$

$$= 5$$

In this case the expression $b^2 - 4ac$ under the radical turned out to be zero. The equation has only one root.

EXAMPLE 4:

Find the roots of the equation

$$2x^2 + 4x + 3 = 0$$

SOLUTION:

a = 2, b = 4 and c = 3. Therefore

$$x = \frac{-4 \pm \sqrt{4^2 - (4 \cdot 2 \cdot 3)}}{2 \cdot 2}$$

$$= \frac{-4 \pm \sqrt{-8}}{4}$$

In this case the expression $b^2 - 4ac$ under the radical turned out to be a negative number (-8). Since the square of a real number cannot be negative, there is no real number equal to $\sqrt{-8}$. Consequently the expression for x above cannot be equal to any real number. In this case, the quadratic equation had no real roots.

What is it that determines whether a quadratic equation has two roots, one root, or no roots? Looking back at the examples you will see that everything depends on the expression $b^2 - 4ac$, called the discriminant.

1.	If $b^2 - 4ac > 0$, there are two real roots.
2.	If $b^2 - 4ac = 0$, there is one real root.
3.	If $b^2 - 4ac < 0$, there are no real roots.

PROBLEM SET 8:

In each of the following cases give the value of the discriminant $b^2 - 4ac$ and state the number of real roots which the equation will have.

1. $x^2 - 5x + 6 = 0$
2. $3x^2 + 2x + 4 = 0$
3. $x^2 + x + 1 = 0$
4. $x^2 - 6x + 9 = 0$
5. $25x^2 + 10x + 1 = 0$
6. $2x^2 + 9x - 4 = 0$
7. $3x^2 - 4x - 2 = 0$
8. $x^2 - 9x + \frac{81}{4} = 0$
9. $15x^2 + 5x + 1 = 0$
10. $x^2 + 8x + 4 = 0$

In each of the following cases give the roots of the quadratic, or state that there are no real roots if that is the case. Express all roots in decimal form, referring to your square root table when needed.

11. $x^2 - 3x - 4 = 0$
12. $x^2 + 13x + 40 = 0$
13. $2x^2 + x + 11 = 0$
14. $6x^2 - 5x + 1 = 0$
15. $5x^2 + 9x + 4 = 0$
16. $x^2 + 5x + 1 = 0$
17. $x^2 + 2x - 4 = 0$
18. $x^2 + 7x - 1 = 0$
19. $2x^2 + 11x - 3 = 0$

REVIEW PROBLEM SET 9:

1. Complete the square to put these quadratic functions in the form $y = a(x - k)^2 + p$.

   a. $y = 2x^2 + 8$

   b. $y = -3x^2 + 2x - 2$

   c. $y = x^2 - \frac{7}{2}x + 3$

   d. $y = -2x^2 - 5x - 1$

2. Draw the graphs of the functions in Problem 1. Show the vertex, axis of symmetry and y-intercept in each case.

3. In each of the following cases, give the equation for the quadratic function, or write "none" if no such quadratic function exists.

   a. The parabola opens downward and has vertex (-1, -9). It is congruent to $y = 6x^2$.

   b. The parabola opens downward, has axis of symmetry $x = 3$ and vertex (3, 2). It has no x-intercepts.

   c. The parabola opens downward and is congruent to $y = -5x^2$. The axis of symmetry is $x = 0$ and the y-intercept is 11.

4. Use the Square Root Table to find the following square roots.

   a. $\sqrt{666}$
   b. $\sqrt{1280}$
   c. $\sqrt{5000}$
   d. $\sqrt{.78}$
   e. $\sqrt{9.3}$
   f. $\sqrt{83.4}$

5. Solve the following quadratic equations by square completion. If there are no roots, write "none."

   a. $x^2 - 3x - 18 = 0$
   b. $x^2 - 10x - 11 = 0$
   c. $5x^2 - 5x - 10 = 0$
   d. $x^2 + 2x + 6 = 0$
   e. $x^2 + 8x + 16 = 0$
   f. $x^2 - 2x - 16 = 0$

6. Solve by the quadratic formula.

   a. $5x^2 - 4x - 1 = 0$
   b. $x^2 + 2x - 4 = 0$
   c. $x^2 - 4x - 12 = 0$

7. How many real roots does each equation have? Use the discriminant to decide.

a. $x^2 + 7x + 5 = 0$ b. $x^2 - x + 6 = 0$ c. $x^2 - x + \frac{1}{4} = 0$

SECTION 10: SCIENTIFIC NOTATION AND MOLARITY

10-1 Review of Scientific Notation

In the coming lesson sequence on applied quadratic equations, you will be studying the concentrations of various ions in the digestive system. Such concentrations are often very small. For example, the hydrogen ion concentration $[H^+]$ in the colon might be .0000002 moles per liter (corresponding to pH = 6.7). Such a number is awkward to write, and it is easy to make an error and put down the wrong number of zeros. For these reasons we will often use scientific notation in our calculations.

EXAMPLE:

Write .0000002 in scientific notation.

SOLUTION:

We move the decimal point 7 places to the right and compensate by multiplying by 10^{-7} to get

$$\underbrace{.0000002}_{\text{7 places}} = 2 \times 10^{-7}$$

The following example illustrates how a very large number may be written in scientific notation.

EXAMPLE:

There are about

602,300,000,000,000,000,000,000

molecules in a mole of any substance. We can express this number in scientific notation by moving the decimal point 23 places to the left and multiplying by 10^{23}; $6.023 \times 10^{23} = \underbrace{602,300,000,000,000,000,000,000.}_{\text{23 places}}$

You may recognize this number as "Avogadro's number."

We say that numbers like 82×10^{12} or 1×10^{-7}, which are written as some number times a power of 10, are in scientific notation. We say that there is always more than one way to write a number in scientific notation. For example, 237,000,000 might be written as 237×10^6, 23.7×10^7, 2.37×10^8 or $.237 \times 10^9$.

Our definition is looser than others you will encounter outside of this course. For example it is common to require that the number multiplied by the power of ten be between 1 and 10. We are using a looser definition here because it simplifies adding and subtracting numbers written in scientific notation.

10-2 Concentration and Molarity

In the upcoming sequence on chemical equilibrium, we will be much concerned with the concentrations of molecules and ions in solution. Chemists use the symbol [] for

concentration. For example, the concentration of hydrogen ions is written $[H^+]$, the concentration of carbonate ions is written $[CO_3^{--}]$ and so forth.

Also notice that our notation for describing ions in solution (e.g., CO_3^{--} differs from what you have used in science (e.g., CO_3^{-2}). Both notations are in common use, and you should become familiar with both.

We could state these concentrations in many different ways--for example, in tons of substance per cubic mile of solution, pounds per gallon, or milligrams per milliliter. However, we will always state our concentrations as molarities--that is, moles of solute per liter of solution. And we will use the captial letter "M" as an abbreviation for "molar."

To make up one liter of a 1-molar solution of sucrose in water (that is, [sucrose] = 1 M) we take one mole of sucrose and add enough water to make up one liter of solution. The trick is to know how much sucrose makes up a mole. You may remember that one mole of anything is simply 6.023×10^{23} of them. Therefore, one way to put together one mole of sucrose would be to count out 6.023×10^{23} molecules. However, chemists long ago tired of counting molecules whenever they needed to make molar solutions. They discovered a shortcut. The found (don't ask how!) that an amount in grams of a substance which is equal to its molecular weight would always contain 6.023×10^{23} molecules. For example, to come up with 6.023×10^{23} molecules of sucrose, we first find the molecular weight. Second, we weigh out a mass of sucrose in grams which is numerically equal to its molecular weight. We illustrate this process below.

1. Determine the approximate molecular weight of sucrose.

element	Atomic Weight
Carbon (C)	12
Hydrogen (H)	1
Oxygen (O)	16

sucrose $C_{12}H_{22}O_{11}$

$C_{12} \longrightarrow 12 \times 12 = 144$

$H_{22} \longrightarrow 22 \times 1 = 22$

$O_{11} \longrightarrow 11 \times 16 = +176$

Molecular weight 342 amu

2. Weigh out 342 grams of sucrose to obtain 6.023×10^{23} molecules of sucrose.

In reality, the idea of gram molecular weight ("mole" in other words) was used before the number 6.023×10^{23} was determined. However, it is perhaps easier to think of a mole as a fixed number of molecules and the gram molecular weight procedure as a method for getting this number.

In mathematics we will not be too concerned with the procedure for determining the molecular weights of different compounds. However we will repeatedly use the idea that 1 mole contains 6.023×10^{23} things. For example, to make up one liter of a 1-molar solution of bowling balls, we would take 6.023×10^{23} bowling balls, and add enough water to make one liter of solution. (6.023×10^{23} bowling balls would cover the entire earth to a depth of about 3,500 miles!)

PROBLEM SET 10:

Express the following numbers in scientific notation.

1. 6,000,000,000
2. .0000000009
3. .00010004
4. 6,473,296,734
5. 45
6. 403,000,000,000,000
7. .0010000007
8. 10,470
9. 63,400,000
10. .000006402

Convert the following numbers to standard decimal form.

11. 16×10^{10}
12. $.4 \times 10^{-12}$
13. $3.141592654 \times 10^{6}$
14. $314159265.4 \quad 10^{-6}$
15. 6.023×10^{12}
16. 780×10^{-5}
17. 4.8×10^{-7}
18. $.0000005 \times 10^{17}$
19. $.0007 \times 10^{-7}$
20. 700×10^{3}

Write "T" for true or "F" for false for Problems 21 through 28.

21. A one-molar solution of sucrose means that there will be one mole of sucrose molecules per liter of <u>water</u>.

22. A one-molar solution of sucrose means that there will be one mole of sucrose molecules in a liter of <u>solution</u>.

23. When chemists use the word "concentration" they are referring to the effort necessary to arrive at a solution.

24. The symbol $[Cl^{-}]$ refers to the concentration of chloride ions in a solution.

25. When chemists speak of a "mole" they are referring to a pointy-nosed furry creature that makes tunnels in the earth.

26. An "8 M" solution refers to a solution that has 8 moles of solute in it.

27. The interpretation of

$$[Ba^{++}] = .1 \text{ M}$$

is, "The concentration of barium ions is one-tenth of a mole in a ml of solution."

28. A mole of water has approximately 6.023×10^{-23} molecules.

The following problems are <u>not</u> true or false.

29. How many things are in a mole?

30. Write an English translation of the statement

$$[CO_3^{--}] = .3 \text{ M}$$

31. Describe a practical method of obtaining 6.023×10^{23} molecules of a given pure solid or liquid substance. (Assume that the molecular structure of the substance is known.)

32. How many bumblebees are needed to make up 1 mole of them?

33. The human body contains about 125,000,000,000,000 cells. Write this number in scientific notation.

34. The concentration of erythrocytes (red blood cells) in human blood is about 5,400,000,000 $\frac{\text{erythrocytes}}{\text{milliliter blood}}$. Write this number in scientific notation.

SECTION 11: A REVIEW OF MULTIPLICATION AND DIVISION IN SCIENTIFIC NOTATION AND THE pH SCALE

11-1 Multiplication and Division in Scientific Notation (See Unit I, Section 28)

Suppose we want to find the number of H^+ ions in one liter of a particular solution. We know that in this particular solution there are 2×10^{-7} moles of H^+ per liter, and we knew that there are about 6.023×10^{23} ions per mole of H^+. Therefore,

$$(2 \times 10^{-7} \frac{\text{moles of } H^+}{\text{liter of solution}})(6.023 \times 10^{23} \frac{H^+}{\text{mole } H^+}) = \frac{\text{number of } H^+ \text{ ions}}{\text{liter of solution}}$$

Both numbers are written in scientific notation; that is, each is expressed as a number multiplied by a power of ten. Next we rewrite the above product in an equivalent form, neglecting units.

$$(2 \times 6.023) \times (10^{-7} \times 10^{23})$$

To calculate the product (2×6.023) is easy. It is 12.046. To calculate the product $(10^{-7} \times 10^{23})$ we first recall a basic property of exponents, the additive property of exponents. Let x be a positive number and a and b be any real numbers. Then

$$(x^a)(x^b) = x^{a+b}$$

Returning to the problem above,

$$10^{-7} \times 10^{23} = 10^{16}$$

Now we have the answer to our original problem,

$$(2 \times 10^{-7})(6.023 \times 10^{23}) = 12.046 \times 10^{16}$$

and we see that even though the $[H^+]$ is a small number (2×10^{-7}), the number of H^+ ions in a liter of this solution is very large (12.046×10^{16}).

The technique used above shows how to multiply two numbers in scientific notation. Two more examples follow.

EXAMPLE:

Find $(3 \times 10^{23}) \times (8 \times 10^{-30})$

SOLUTION:

We have $3 \times 8 = 24$ and also

$$10^{23} \times 10^{-30} = 10^{23+(-30)}$$
$$= 10^{-7}$$

Therefore the product is

$$24 \times 10^{-7}$$

EXAMPLE:

Find $(1.2 \times 10^5) \times (2 \times 10^{-5})$

SOLUTION:

$1.2 \times 2 = 2.4$ and also

$$10^5 \times 10^{-5} = 10^{5+(-5)}$$
$$= 10^0$$
$$= 1$$

The answer is therefore simply 2.4.

We conclude with a division example. When two powers are divided, the rule

$$\frac{x^a}{x^b} = x^{a-b}$$

applies. We call this the subtractive property of exponents in division.

EXAMPLE:

Find the number of km^2 covered by 6×10^{23} bowling balls in a single layer. Assume 2×10^7 bowling balls/km^2. (BB = Bowling Balls)

SOLUTION:

First we invert and multiply

$$6 \times 10^{23} \text{ BB} \cdot \frac{1 \text{ km}^2}{2 \times 10^7 \text{ BB}} = \frac{6}{2} \times \frac{10^{23}}{10^7} \text{ km}^2$$
$$= 3 \times 10^{(23 - 7)} \text{ km}^2$$

Which reduces to,

$$3 \times 10^{16} \text{ km}^2$$

11-2 pH and Molarity (See Unit I, Section 29)

The concentration of hydrogen ions in solution is so important that a special scale, the pH scale, has been created to measure it. If we write $[H^+]$ in the form

$$[H^+] = 1 \times 10^{-n} \text{ M},$$

then

$$\text{pH} = n \text{ (by definition)}$$

This is trickier than it looks at first glance. For example, suppose

$$[H^+] = 5 \times 10^{-4} \text{ M},$$

what is the pH? Since the power of ten is multiplied by a five instead of a one, the answer is not immediately obvious. (The answer is not 4.) However, we can transform the number 5×10^{-4} into the form 1×10^{-n} by using information on decimal powers of ten previously discussed in Unit I, Section 29. The decimal power of ten table is repeated at the end of this section. In it we find that

$$5 = 10^{.7}$$

When $10^{.7}$ is substituted for 5 in the above equation we get

$$5 \times 10^{-4} \approx 10^{.7} \times 10^{-4}$$

$$\approx 10^{(.7 - 4)}$$

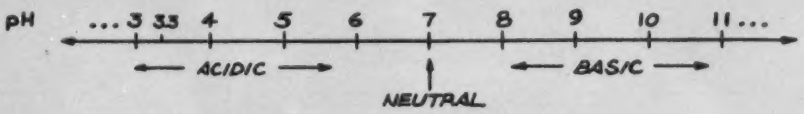
$$\approx 10^{-3.3}$$

and we can find the pH.

$$[H^+] \approx 1 \times 10^{-3.3} \text{ M}$$

$$\text{pH} \approx 3.3$$

The pH scale exists mainly to describe the acidic or basic properties of a given solution. The scale below shows the relation between pH, acidity and basicity. Notice that a pH of 3.3 is an acidic pH.



For convenience, we repeat the decimal power of ten table.

x	10^x	x	10^x	x	10^x
0	1.0	.4	2.5	.8	6.3
.1	1.25	.5	3.2	.9	7.9
.2	1.6	.6	4.0	1.0	10.0
.3	2.0	.7	5.0		

PROBLEM SET 11:

1. a. Write a mathematical equation that describes the <u>additive property of exponents</u> in multiplication.

   b. Do the same for the <u>subtractive property of exponents</u> in division.

Perform the following multiplications and divisions.

2. $(6 \times 10^{23})(3 \times 10^{-7})$
3. $(6 \times 10^{23})(4 \times 10^{-8})$
4. $(4 \times 10^{-10})(3 \times 10^{-9})$
5. $(5 \times 10^{20})(7 \times 10^{6})$
6. $(16 \times 10^{-8})(.5 \times 10^{-12})$
7. $(48 \times 10^{4})(.1 \times 10^{-3})$
8. $\dfrac{40 \times 10^{16}}{8 \times 10^{2}}$
9. $\dfrac{40 \times 10^{10}}{5 \times 10^{-2}}$
10. $\dfrac{45 \times 10^{-6}}{5 \times 10^{-3}}$
11. $\dfrac{35 \times 10^{6}}{7 \times 10^{2}}$
12. $\dfrac{24 \times 10^{-3}}{6 \times 10^{-7}}$

13. a. How many H^+ ions are in 1 liter of a solution where $[H^+] = 4 \times 10^{-6}$ M? (Let Avogadro's number = 6×10^{23})

    b. A particular extremely basic solution has a $[H^+] = 6 \times 10^{-14}$ M. How many H^+ ions are in one liter of this solution?

    c. How many are in 1 ml?

Calculate the number of H^+ ions in one liter for each of the following H^+ concentrations. (Use Avogadro's Number = 6×10^{23})

14. $[H^+] = 1$ M

15. $[H^+] = 5 \times 10^{-2}$ M

16. $[H^+] = 3 \times 10^{-4}$ M

17. $[H^+] = 2 \times 10^{-5}$ M

18. $[H^+] = 1 \times 10^{-7}$ M

19. $[H^+] = 4 \times 10^{-8}$ M

20. $[H^+] = 8 \times 10^{-10}$ M

21. pH = 2 means $[H^+] = 1 \times 10^{-2}$ M

    pH = 3 means $[H^+]$ = ?

State the $[H^+]$ for each of the following pH's.

22. pH = 4

23. pH = 7

24. pH = 9

25. pH = 11

26. $[H^+] = 1 \times 10^{-5}$ M means pH = 5

    $[H^+] = 1 \times 10^{-6}$ M means pH = ?

State the pH of each of the following $[H^+]$'s.

27. $[H^+] = 1 \times 10^{-8}$ M

28. $[H^+] = 1 \times 10^{-10}$ M

29. $[H^+] = 1 \times 10^{-2}$ M

30. $[H^+] = 1 \times 10^{-1}$ M

31. a. pH < 7 means the solution is _____.

    b. pH > 7 means the solution is _____.

32. a. pH = 4.3 means $[H^+] = 1 \times 10^{x}$ x = ?

    b. $10^{-4.3} = 10^{y} \times 10^{-5}$ y = ?

    c. $10^{.7} = n$ n = ?

    d. State the $[H^+]$ in scientific notation.

Convert the following pH's to $[H^+]$'s in the form $x \times 10^{-n}$ M, where n is a whole number. Follow the pattern in Problem 32.

33. pH = 1.2

34. pH = 3.1

35. pH = 4.7

36. pH = 5.1

37. pH = 5.8

38. pH = 7.5

39. pH = 8.9

40. pH = 10.3

41. There are 5.4×10^{9} erythrocytes/ml blood. An average adult male has about 5.5 liters of blood. Recall that there are 10^{3} milliliters in a liter. Show calculations verifying that there are 2.97×10^{13} erythrocytes in an average adult male body.

42. Each erythrocyte contains about 270,000,000 hemoglobin molecules. Put this number into scientific notation.

43. Show calculations leading to a figure of 8.019×10^{21} for the number of hemoglobin molecules in an adult male. (You will need to refer to Problems 41 and 42.)

44. About 2,300,000 erythrocytes die each second and must be replaced at the same rate from the bone marrow, where erythrocytes are produced. Put this number into scientific notation.

45. There about 1.8×10^4 days in a period of 50 years. How many erythrocytes must be produced by the bone marrow in a 50-year period? (You will have to supply the conversion factor for seconds per day.)

46. The diameter of an erythrocyte is about 7 micrometers. A micrometer is a thousandth of a millimeter. Express the diameter of an erythrocyte in millimeters and give the answer in scientific notation.

47. The thickness of an average cell membrane in the human body is 10 nanometers. There are one thousand nanometers in a micrometer. Express the thickness of a cell membrane in millimeters, and give the answer in scientific notation.

SECTION 12: EQUILIBRIUM AND SOLUBILITY

12-1 What Lies Ahead?

This section begins a mathematical treatment of certain chemical equilibria. We are examining chemical equilibria for two reasons. First, many equilibrium problems require the use of the quadratic formula. Second, many chemical equilibria are important in biomedicine. The following are some examples which we will investigate.

1. The behavior of antacids in the digestive tract.
2. The behavior of certain food additives in the digestive tract.
3. The behavior of heavy-metal paint pigments after ingestion.

Chemical equilibrium is also important in the blood. However, we won't investigate blood-related equilibria because the associated mathematics tends to be more difficult.

Since chemical equilibrium is the focus of this applied math sequence, the following section reviews certain basic ideas relating to equilibrium. The material was originally presented in Science Unit II.

12-2 Equilibrium

In physics, we say an object is in equilibrium if it is not subject to any unbalanced forces. Such objects often just sit in one spot, like a sack of potatoes resting on the kitchen floor. This is an example of a static physical equilibrium. An airplane in level flight is an example of a dynamic equilibrium, in other words, an equilibrium associated with motion. The forward motion of the airplane produces an upward force on the wings which counterbalances the downward force of gravity. The airplane moves neither up or down; therefore it is in equilibrium vertically.

> Chemical equilibrium is dynamic. It occurs when two opposite reactions take place at the same rate and at the same time.

For example, consider a saturated solution of $CaSO_4$ (calcium sulfate--better known as "plaster of paris"). A saturated solution of $CaSO_4$ is one in which the solvent (water) contains as much of the dissolved solute (calcium sulfate) as it can hold. If there is excess calcium sulfate, it will be found lying at the bottom of the beaker. But chemical changes are still going on. Some of the molecules of crystalline (solid) $CaSO_4$ are dissolving into the solution; some of the molecules in the solution are precipitating out. (See Figure 1, right.)

When these two effects exactly counteract each other--that is, when they occur at the same rate--then we have equilibrium. The solution remains saturated, and there is no net change in the concentration of the solution. Remember this: a saturated solution is in a state of chemical equilibrium.

Another example of this kind of equilibrium is shown in Figure 2 (below right). Here, we have a box divided by a partition with a hole in it. Gas molecules are zipping about in random motion on both sides of the partition.

In Part A of Figure 2, there are more gas molecules on the left side of the partition than on the right. Therefore, more molecules will move through the hole from left to right than in opposite direction. This is a non-equilibrium condition, which we symbolize with the unequal arrows below the box.

The molecules will continue this unequal flow until there are as many on the right side of the partition as on the left. Part B of Figure 2 shows this situation.

Now we have equilibrium, as shown by the equal arrows below the box. The molecules have not stopped moving between the two sides of the box, but they are moving in equal numbers in each direction--in other words, the equilibrium is dynamic.

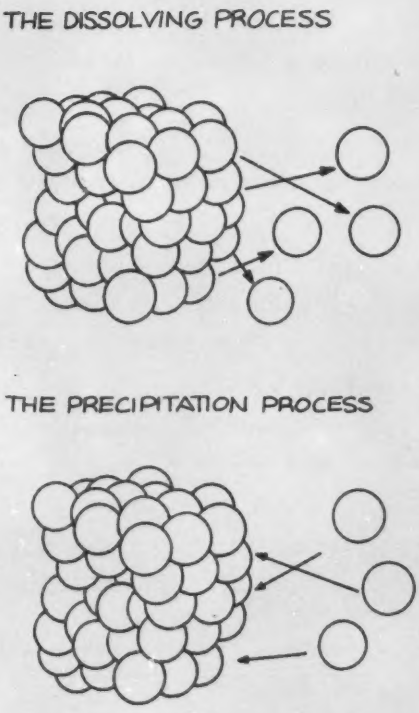


FIGURE 1

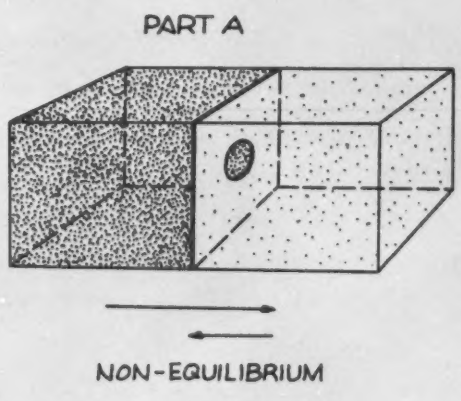


FIGURE 2

12-3 Solubility Products

Let's return to our saturated solution of $CaSO_4$ and water. Since the solution is saturated, the amount of $CaSO_4$ dissolved in the solution will not increase, no matter how much solid $CaSO_4$ we may add. This is because there is a limit to the amount of $CaSO_4$ that will dissolve in a given amount of water, at a given temperature.

Chemists represent the equilibrium of $CaSO_4$ in water by the chemical equation

$$CaSO_4\downarrow \rightleftharpoons Ca^{++} + SO_4^{--}$$

Here the symbol $\rightleftharpoons$ indicates equilibrium. That is, the $CaSO_4$ molecules are dissociating into Ca^{++} ions and SO_4^{--} ions at the same rate that these ions are recombining to form $CaSO_4$. When a Ca^{++} ion and a SO_4^{--} ion combine, a $CaSO_4$ molecule precipitates out of the solution. The downward arrow ($\downarrow$) indicates this.

There is a numerical relationship which governs the concentrations of Ca^{++} and SO_4^{--} in the solution. This relation is

$$2 \times 10^{-5} = [Ca^{++}] \times [SO_4^{--}]$$

The number 2×10^{-5} is called the solubility product of the solution at equilibrium. Chemists use the symbol K_{sp} for solubility product.

Each of the chemical solutions we will study has a different K_{sp}--a different value for its solubility product.

The solubility relation tells us two things.

1. $[Ca^{++}] \times [SO_4^{--}] < 2 \times 10^{-5}$ indicates that the solution is less than saturated. The $CaSO_4$ is completely dissolved, and more can be added. If we add more $CaSO_4$, enough will dissolve to raise the concentrations of Ca^{++} and SO_4^{--} until $[Ca^{++}] \times [SO_4^{--}] = 2 \times 10^{-5}$.

2. $[Ca^{++}] \times [SO_4^{--}] > 2 \times 10^{-5}$ indicates that a precipitate will form (under ordinary conditions). As the Ca^{++} and SO_4^{--} combine to form this precipitate, their concentrations go down, until $[Ca^{++}] \times [SO_4^{--}] = 2 \times 10^{-5}$.

12-4 Our Attitude Toward Chemical Equilibrium

In this mathematics course we are primarily concerned with the mathematics of equilibrium. We are not concerned with many of the purely chemical aspects of the subject.

So, we will not worry about why the solubility product $[Ca^{++}] \times [SO_4^{--}]$ is a constant for a saturated solution, and why that constant happens to be 2×10^{-5}. The numbers we use will come from the literature of chemistry. Also, we won't bother about why $CaSO_4$ dissociates into Ca^{++} and SO_4^{--} ions in solution and similar chemical questions.

12-5 A Solubility Problem

Suppose we mix a solution containing K^+ ion with a solution containing acid tartrate ion, $HC_4H_4O_6^-$ (abbreviated $HTar^-$). Given the following concentrations of these solutions immediately after mixing, will a precipitate form?

$[K^+] = 2 \times 10^{-2}$ M

$[HTar^-] = 3 \times 10^{-2}$ M

and K_{sp} for KHTar = 3.8×10^{-4}

Figure 3 to the right shows this situation.

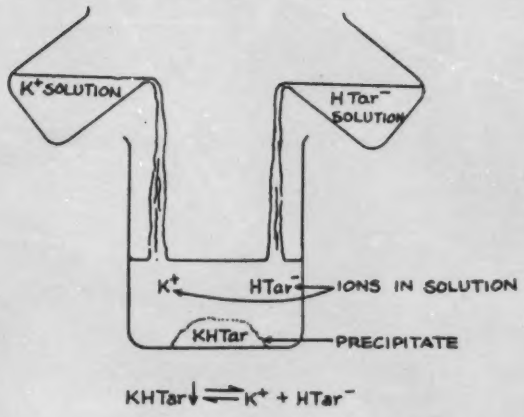


FIGURE 3

Immediately after mixing,

$$[K^+] \times [HTar^-] = (2 \times 10^{-2}) \times (3 \times 10^{-2})$$
$$= 6 \times 10^{-4}$$

But $K_{sp} = 3.8 \times 10^{-4}$, so that

$$[K^+] \times [HTar^-] > K_{sp}$$

Therefore, a precipitate will form. The precipitation of KHTar reduces both the $[K^+]$ and $[HTar^-]$ in solution. When the $[K^+]$ and $[HTar^-]$ are reduced to the level that $[K^+] \times [HTar^-] = 3.8 \times 10^{-4}$, then the $[K^+]$, $[HTar^-]$ and the amount of precipitate will stabilize at constant levels.

12-6 Comparing Sizes of Numbers in Scientific Notation

In the previous section we had to compare the sizes of two numbers in scientific notation. For the particular example given, it was easy, but it won't always be so easy. In the material which follows you will repeatedly be required to compare the sizes of two numbers in scientific notation. Therefore, we'll provide some practice in this important mathematical skill now, so you'll be better prepared later.

EXAMPLE:

Is

$$6300 \times 10^{-14} > .34 \times 10^{-12} \text{ ?}$$

SOLUTION:

What makes this problem tricky is that the two powers of ten involved aren't equal. This suggests the first step--rewrite one of the numbers so that both will have the same power of ten. We could rewrite either number, but since we have to rewrite only one, we choose 6300×10^{-14} and note that

$$10^{-14} = 10^{-2} \times 10^{-12}$$

When $(10^{-2} \times 10^{-12})$ is substituted for 10^{-14}, then

$$6300 \times 10^{-14} = 6300 \times (10^{-2} \times 10^{-12})$$

After rearranging the parentheses,

$$6300 \times 10^{-14} = (6300 \times 10^{-2}) \times 10^{-12}$$
$$= 63 \times 10^{-12}$$

Now that both numbers are expressed as a number times 10^{-12}, they may be compared easily.

$$63 \times 10^{-12} > .34 \times 10^{-12}$$

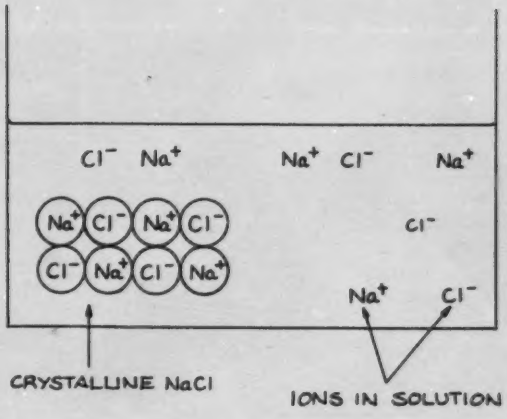
because

$$63 > .34$$

There is one more important point that needs to be made. Even though both 63×10^{-12} and $.34 \times 10^{-12}$ are extremely small numbers, 63×10^{-12} is more than 100 times larger than $.34 \times 10^{-12}$. Similar observations will become crucial later on when we add and subtract numbers in scientific notation.

PROBLEM SET 12:

The diagram to the right represents a saturated solution of NaCl in a beaker. The phrases "saturated solution" and "equilibrium solution" are synonymous. Determine whether the following statements are true or false.



1. None of the NaCl in crystalline form will ever dissolve as ions.

2. None of the Na^+ ions will ever crystallize back into solid form.

3. This kind of equilibrium is a dynamic process.

4. If more NaCl is added to the solution, the concentration of Na^+ ions and Cl^- ions in solution will increase.

5. One mole of NaCl has a mass that is equivalent to the gram molecular weight of NaCl.

6. If water were added to the beaker shown in the diagram and time was allowed for the solution to reach equilibrium again, then less NaCl would be in solid form.

7. Assume the conditions of Problem 6. The concentration of the Cl^- ion would be greater at the new equilibrium than it was before the water was added.

Each of the statements numbered 8 through 16 contains an error. Find it and correct it.

8. The symbol "$[Ca^{++}]$" is read, "The density of calcium ion."

9. If the product of $[Ca^{++}] \times [SO_4^{--}]$ is greater than the solubility product, then all of the $CaSO_4$ is dissolved.

10. The product $[Ca^{++}] \times [SO_4^{--}]$ is an example of a dissolubility product.

11. The symbol K_{sp} is an abbreviation for the phrase, "catsup."

12. A saturated solution is in equilimbo.

13. The chemical equation

$$CaSO_4\downarrow \rightleftharpoons Ca^{++} + SO_4^{--}$$

signifies that the reaction is irreversible.

14. $(2 \times 10^{-1})(6 \times 10^{-2}) = (12 \times 10^{-12})$

15. The interpretation of $[Ba^{++}] = .1$ M is, "The concentration of barium ions is one-tenth of a mole per ml of solution."

16. A mole of water has approximately 6.023×10^{-23} molecules.

For each pair of numbers below, first rewrite the first term, so that both have the same power of ten. The, put the correct symbol of inequality between them.

EXAMPLE:

14×10^{-8}, $.7 \times 10^{-7}$

$1.4 \times 10^{-7} > .7 \times 10^{-7}$

17. 18×10^{-9}, $.9 \times 10^{-7}$

18. 47×10^{-10}, 4.1×10^{-8}

19. $.61 \times 10^{-4}$, 18×10^{-3}

20. 7×10^{-10}, $.8 \times 10^{-8}$

21. $.027 \times 10^{-5}$, 30×10^{-8}

22. 38×10^{-10}, 9×10^{-8}

23. $.3 \times 10^{-2}$, $.005 \times 10^{-3}$

24. 6×10^{-5}, 140×10^{-6}

25. 2.1×10^{-2}, 1.9×10^{-4}

26. 43×10^{-10}, $.2 \times 10^{-11}$

27. Identify the number pairs above where one number is at least 100 times greater than the other.

28. A solution containing a high concentration of Ca^{++} ion is poured into a solution of H_2SO_4. Immediately after the solutions are mixed the concentration of Ca^{++} ion is .1 M. The concentration of SO_4^{--} ion is .2 M.

a. What is the initial concentration of Ca^{++} ion?

b. What is the initial concentration of SO_4^{--} ion?

c. Calculate the initial value of the product $[Ca^{++}] \times [SO_4^{--}]$.

d. At equilibrium the product is $[Ca^{++}] \times [SO_4^{--}] \approx 2 \times 10^{-5}$. Is the product found in Part c greater than or less than 2×10^{-5}?

e. Will a precipitate form?

29. A solution containing Ba^{++} ions is mixed with one containing SO_4^{--} ions. Initially in the resulting solution, $[Ba^{++}] = 1 \times 10^{-8}$ M and $[SO_4^{--}] = .1$ M. The K_{sp} for $BaSO_4$ is approximately 1×10^{-10}.

a. What is the initial concentration of Ba^{++} ion in the resulting solution?

b. What is the initial concentration of SO_4^{--} ion in the resulting solution?

c. What is the product of the initial concentrations?

d. Will a precipitate of $BaSO_4$ form?

30. Two solutions are mixed. One contains Ag^+ ions and one contains Br^- ions. Immediately after mixing the two solutions, $[Ag^+] = 3 \times 10^{-7}$ M and $[Br^-] = 2 \times 10^{-8}$ M. The K_{sp} for AgBr is approximately 4×10^{-13}. Show calculations that confirm that a precipitate will not form.

SECTION 13: APPLIED QUADRATIC EQUATIONS

13-1 Some Slightly Soluble Substances

During the next few sections we will take a closer look at chemical equilibrium, starting with the solubilities of some slightly soluble chemicals. Table 1 below lists some biomedically important chemicals we will study.

Antacids	$Al(OH)_3$	Aluminum Hydroxide Gel
	$Mg(OH)_2$	Milk of Magnesia
	$CaCO_3$	Calcium Carbonate (chalk)
	$AlPO_4$	Aluminum Phosphate Gel
Food Additives	$CaC_4H_4O_6$	Calcium Tartrate
	$KHC_4H_4O_6$	Potassium Acid Tartrate
	$CaCO_3$	Calcium Carbonate
Paint Pigments	$PbCO_3$	"White Lead"
	$PbSO_4$	"White Lead"
	$Pb(OH)_2$	"White Lead"
	$PbCrO_4$	Chrome yellow
	HgS	Vermillion
	CdS	Cadmium Red

TABLE 1

Partial solubility is supposed to be an advantage in an antacid. (If the antacid doesn't dissolve very well, it will remain effective longer, right?) However, the calcium and aluminum salt antacids have an unfortunate side-effect--they tend to cause constipation. For example, when a molecule of calcium carbonate precipitates out of solution, it takes six molecules of water with it. Too much of this sort of thing and the feces become too dry (constipation). The food additives listed also share this side-effect.

The paint pigments listed in Table 1 are all important poisons. The lead (Pb) group is particularly important. Three different lead salts have the same common name, "White lead," and are found in many white and yellow paints. The ingestion of lead-based paint chips by children from six months to two years old poses a serious public health problem in this country. It occurs most frequently among children who are living in older homes that have been allowed to run down.

Undissolved lead is harmless. It is the dissociated lead ion (Pb^{++}) that is harmful. Once the Pb^{++} ion is released it can be absorbed into the intestine and react with chemicals in the body in undesirable ways.

The following problem demonstrates how to calculate equilibrium concentrations using solubility products.

PROBLEM:

Enough potassium acid tartrate is added to water to establish equilibrium. What is the equilibrium concentration of K^+?

SOLUTION:

Let KHTar be an abbreviation for $KHC_4H_4O_6$. The chemical reaction at equilibrium is

$$KHTar \rightleftharpoons K^+ + HTar^-$$

So the equilibrium equation is

$$K_{sp} = [K^+] \times [HTar^-]$$

In this instance, K_{sp} is 3.8×10^{-4}.

When we first add the KHTar, it is all solid--none has dissolved--thus the initial concentrations $[K^+]$ and $[HTar^-]$ are both zero.

Let z be the equilibrium concentration of K^+. The only source of $HTar^-$ ions is from the dissolution of KHTar, and one $HTar^-$ is formed for each K^+. Therefore, the equilibrium concentration of $HTar^-$ must also be z. We summarize this in Table 2 below.

TABLE 2

		IONS	
		K^+	$HTar^-$
Concentrations in M	Initially	0	0
	At Equilibrium	z	z

Now we return to the equilibrium equation

$$K_{sp} = [K^+] \times [HTar^-]$$

This becomes

$$3.8 \times 10^{-4} = z \cdot z$$

$$3.8 \times 10^{-4} = z^2$$

This is our first applied quadratic equation. Its solution is

$$z = \pm\sqrt{3.8 \times 10^{-4}}$$

What are we to make of $z = -\sqrt{3.8 \times 10^{-4}}$? Could $[K^+]$ really be negative? Apparently, this is a solution to our quadratic equation which makes mathematical sense, but does not make physical (or chemical) sense. From now on we will ignore all solutions to equations which lead to negative concentrations since negative concentrations are meaningless. Then we are left with

$$z = +\sqrt{3.8 \times 10^{-4}}$$

This equation suggests the need to take the square root of a number in scientific notation. This is the topic of the next subsection. Meanwhile, we state without explanation

$$z = 1.95 \times 10^{-2}$$

In other words, the equilibrium concentrations of both K^+ and $HTar^-$ are 1.95×10^{-2} M.

13-2 Finding the Square Root of a Number in Scientific Notation

You probably recall some of the techniques for simplifying radical expressions. One technique is the removal of perfect square factors. For example, suppose you need to find $\sqrt{8}$. Eight is not a perfect square but 8 does have a perfect square factor, namely 4. The perfect square factor 4 can be "removed" as follows.

$$\sqrt{8} = \sqrt{4 \cdot 2}$$

$$= \sqrt{4} \cdot \sqrt{2}$$

$$= 2\sqrt{2}$$

Taking another example, suppose we want to find $\sqrt{363}$. We notice that $363 = (121) \times (3)$ and $121 = 11^2$. Therefore,

$$\sqrt{363} = \sqrt{121 \cdot 3}$$

$$= \sqrt{121} \cdot \sqrt{3}$$

$$= 11\sqrt{3}$$

If we didn't have a square root table it would be easier to multiply 11 times $\sqrt{3}$ than to calculate $\sqrt{363}$. This is why the process is called the "simplification" of radicals.

If you properly apply the preceding technique you can find the square root of a number in scientific notation. Since numbers in scientific notation contain powers of 10, we will begin with the following question. When is a power of 10 a perfect square? The following multiplications help lead to an answer.

$$10 \times 10 = 10^2$$

$$10^2 \times 10^2 = 10^4$$

$$10^3 \times 10^3 = 10^6$$

$$10^{-1} \times 10^{-1} = 10^{-2}$$

We can deduce the pattern from these examples; when we square a power of 10 we will always get an even power of 10. Every even power of 10 is a perfect square. For example, $\sqrt{10^4} = 10^2$; $\sqrt{10^{-2}} = 10^{-1}$; $\sqrt{10^{64}} = 10^{32}$.

EXAMPLE:

Find $\sqrt{64 \times 10^8}$

SOLUTION:

$$\sqrt{64 \times 10^8} = \sqrt{64} \times \sqrt{10^8}$$

$$= 8 \times 10^4$$

EXAMPLE:

Find $\sqrt{28 \times 10^{-10}}$

SOLUTION:

$$\sqrt{28 \times 10^{-10}} = \sqrt{28} \times \sqrt{10^{-10}}$$

$$= \sqrt{28} \times 10^{-5}$$

Referring to the Square Root Table, we find that $\sqrt{28} \approx 5.292$. Therefore,

$$\sqrt{28 \times 10^{-10}} \approx 5.292 \times 10^{-5}$$

EXAMPLE:

Find $\sqrt{2.5 \times 10^{17}}$

SOLUTION:

This time the power of 10 is not an even number, and therefore 10^{17} is not a perfect square. However, 10^{16} is a perfect square. Therefore, we factor out 10^{16}.

$$\sqrt{2.5 \times 10^{17}} = \sqrt{2.5 \times 10 \times 10^{16}}$$

$$= \sqrt{25} \times \sqrt{10^{16}}$$

$$= 5 \times 10^{8}$$

EXAMPLE:

Find $\sqrt{11 \times 10^{-101}}$

SOLUTION:

Again the exponent of 10 is odd. However $-101 = -102 + 1$ and thus

$$10^{-101} = 10^{-102} \times 10$$

Therefore

$$\sqrt{11 \times 10^{-101}} = \sqrt{11 \times 10 \times 10^{-102}}$$

$$= \sqrt{110} \times \sqrt{10^{-102}}$$

$$= \sqrt{110} \times 10^{-51}$$

Again, referring to the Square Root Table, we find that $\sqrt{110} \approx 10.49$. Therefore,

$$\sqrt{11 \times 10^{-101}} \approx 10.49 \times 10^{-51}$$

PROBLEM SET 13:

1. $\sqrt{16 \times 10^{-2}} =$
2. $\sqrt{25.6 \times 10^{3}} =$
3. $\sqrt{100 \times 10^{-3}} =$
4. $\sqrt{2.5 \times 10^{17}} =$
5. $\sqrt{20 \times 10^{-5}} =$
6. $\sqrt{.81 \times 10^{10}} =$
7. $\sqrt{.169 \times 10^{-7}} =$
8. $\sqrt{400 \times 10^{2}} =$
9. $\sqrt{4.9 \times 10^{-23}} =$
10. $\sqrt{1.21 \times 10^{-76}} =$
11. $\sqrt{90 \times 10^{-155}} =$

12. Calcium tartrate (abbreviated CaTar) has a solubility product of 7.7×10^{-7}. Its dissociation reaction is $CaTar \rightleftharpoons Ca^{++} + Tar^{--}$.

a. Because the ratio of Ca ions to tartrate ions in solid CaTar is one to ____, the number of Ca^{++} ions and Tar^{--} ions in solution will be equal.

b. Based on the statement in Part a, if $[Ca^{++}]$ in the solution is represented by z, then $[Tar^{--}]$ = ?

c. Solve the equation

$$z^2 = 77 \times 10^{-8}$$

for z.

d. What is $[Tar^{--}]$ in a saturated solution of calcium tartrate?

13. Lead chromate ($PbCrO_4$) is a pigment often found in the yellow paint on pencils. It is known as chrome yellow. When $PbCrO_4$ dissolves, it dissociates according to the chemical equilibrium

$$PbCrO_4 \rightleftharpoons Pb^{++} + CrO_4^{--}$$

The concentrations of the ions is ruled by the equation

$$K_{sp} = [Pb^{++}] \cdot [CrO_4^{--}]$$

where $K_{sp} = 1.77 \times 10^{-14}$. What is the concentration of Pb^{++} ions in a saturated solution of chrome yellow? (Include units.)

14. $MgCO_3$ (magnesium carbonate), which is sometimes used as a laxative, has a K_{sp} of about 2.6×10^{-5}. Equal amounts of Mg^{++} and CO_3^{--} ions are formed when $MgCO_3$ dissolves. What is the concentration of Mg^{++} ions for a saturated solution of $MgCO_3$? (Include units.)

15. $CaCO_3$, calcium carbonate, is the chemical name for chalk. It is sometimes used as an antacid. Some authorities even say that it is the best antacid because it doesn't interfere with the body's chemistry. Chalk is only slightly soluble in water. If the product of the molar concentrations of Ca^{++} ions and CO_3^{--} ions in solution exceeds 0.8×10^{-8} at body temperature then $CaCO_3$ precipitates. This sometimes actually occurs in the body. Gallstones are typically about 25% $CaCO_3$. Calculate the $[Ca^{++}]$ in a saturated water solution of chalk.

For Problems 16 through 23, find the equilibrium concentrations of each ion. Assume that excess chemical is available so that the solution is saturated. Round all answers to two digits.

16. Lead Sulfide

$$PbS\downarrow \rightleftharpoons Pb^{++} + S^{--}$$

$$K_{sp} = 3.4 \times 10^{-28}$$

17. Vermillion (a paint pigment)

$$HgS\downarrow \rightleftharpoons Hg^{++} + S^{--}$$

$$K_{sp} = 4 \times 10^{-53}$$

18. Cadmium Sulfide (Aurora Yellow, another paint pigment)

$$CdS\downarrow \rightleftharpoons Cd^{++} + S^{--}$$

$$K_{sp} = 3.6 \times 10^{-29}$$

19. Barium Yellow (Barium Chromate)

$$BaCrO_4\downarrow \rightleftharpoons Ba^{++} + CrO_4^{--}$$

$$K_{sp} = 1.6 \times 10^{-10}$$

20. Barium White (Barium Sulfate)

$$BaSO_4\downarrow \rightleftharpoons Ba^{++} + SO_4^{--}$$

$$K_{sp} = 8.7 \times 10^{-11}$$

21. CoS (Cobalt Sulfide)

$$CoS\downarrow \rightleftharpoons Co^{++} + S^{--}$$

$$K_{sp} = 3 \times 10^{-26}$$

22. Lead Carbonate (one type of White Lead)

$$PbCO_3\downarrow \rightleftharpoons Pb^{++} + CO_3^{--}$$

$$K_{sp} = 3.3 \times 10^{-14}$$

23. Lead Sulfate (another type of White Lead)

$$PbSO_4\downarrow \rightleftharpoons Pb^{++} + SO_4^{--}$$

$$K_{sp} = 1 \times 10^{-8}$$

24. a. The smaller the solubility product, the (more, less) soluble the chemical is.

b. Sulfides generally have relatively (small, large) solubility products and relatively (low, high) solubilities.

c. What was the <u>least</u> soluble compound listed in Problems 16 through 23?

d. What was the <u>most</u> soluble compound?

e. What was the metal (i.e., barium, cadmium, cobalt, lead, mercury) most frequently mentioned in connection with paint pigments?

25. White lead (this time a mixture of $PbCO_3$ and $Pb(OH)_2$) is sometimes used as a ceramic glaze. When the glaze is fired in a kiln, it is transformed into PbO, lead oxide. PbO is insoluble; however, acids such as those found in carbonated beverages will transform PbO back to the potentially dangerous $PbCO_3$. Make a list of at least four beverages that it would be unwise to drink from a ceramic cup that had been glazed with white lead. Incidentally, PbO is <u>yellow</u>.

26. The treatment for lead poisoning is the administration of large amounts of Epsom salt ($MgSO_4 \cdot 7H_2O$.) The primary objective is to void the intestines in the manner of a strong laxative. A secondary effect may be deduced from the solubility product ideas.

a. Suppose a $PbSO_4$ solution is in a state of saturated equilibrium so that

$$[Pb^{++}][SO_4^{--}] = K_{sp}$$

Now suppose that the $[SO_4^{--}]$ is suddenly drastically increased due to the addition of a large amount of $MgSO_4$, a soluble sulfate. Now

$$[Pb^{++}][SO_4^{--}] \text{ will be } (>, =, <)\ K_{sp}.$$

b. What will happen as a result of the change described in Part a?

c. How will this reduce the danger of lead poisoning?

SECTION 14: WEAK ACIDS AND DISSOCIATION CONSTANTS

14-1 Weak Acids and Dissociation Constants

In the last two sections, we saw that the behavior of slightly soluble solids in water is governed by the equilibrium constant K_{sp}. For example, if the solid in question is $CaSO_4$, then the reaction is

$$CaSO_4\downarrow \rightleftharpoons Ca^{++} + SO_4^{--}$$

and $K_{sp} = 2 \times 10^{-5}$. Therefore, the equilibrium equation is

$$2 \times 10^{-5} = [Ca^{++}] \times [SO_4^{--}]$$

But slightly soluble solids are not the only substances governed by equilibrium constants. Weak acids, such as acetic acid (the main constitutent of vinegar), behave in a similar way when dissolved in water. Our mathematical treatment of the behavior of weak acid solutions will enable us to calculate the equilibrium $[H^+]$. This is often important in the laboratory. For example, enzymes are effective only in certain pH ranges. A researcher investigating the behavior of enzymes must be able to make solutions of the proper pH.

Let's take a closer look at the behavior of acetic acid in water. When acetic acid dissolves in water, some molecules dissociate into H^+ and $C_2H_3O_2^-$ ions. Other remain as undissociated $HC_2H_3O_2$ molecules. The equilibrium between dissociated and undissociated $HC_2H_3O_2$ is described by

Acetic Acid $\rightleftharpoons$ Hydrogen Ions + Acetate Ions

$$HC_2H_3O_2 \rightleftharpoons H^+ + C_2H_3O_2^-$$

But here the products on both sides of the equation are soluble in water, unlike the cases we have studied before. This means that the equilibrium equation will be a little different.

The concentrations of $HC_2H_3O_2$, H^+ and $C_2H_3O_2^-$ are governed by the equation

$$K = \frac{[H^+][C_2H_3O_2^-]}{[HC_2H_3O_2]}$$

and K, in this case, is called the dissociation constant. Solubility products and dissociation constants are specialized forms of equilibrium constants.

If certain concentrations are known, it is possible to calculate the value of a particular equilibrium constant.

EXAMPLE:

An acetic acid solution is made so that the initial concentration is 9×10^{-5} M.

After waiting a short while for equilibrium to be established, the pH is measured. It is 4.5.

What is the dissociation constant for acetic acid?

SOLUTION:

First we note that the calculation of the dissociation constant requires us to know three concentrations,

$$\text{i.e.,} \quad [H^+],\ [C_2H_3O_2^-] \quad \text{and} \quad [HC_2H_3O_2]$$

Second we note that the $[H^+]$ is given indirectly in the problem. It may be derived from the pH.

$$pH = 4.5$$

implies that

$$[H^+] = 10^{-4.5}\ M$$

$10^{-4.5}$ may be equivalently written as

$$10^{-4.5} = (10^{.5})(10^{-5})$$

By referring to the decimal powers of ten table in Section 11 we find that

$$10^{.5} = 3.2$$

When 3.2 is substituted for $10^{.5}$ we get

$$[H^+] = 3.2 \times 10^{-5}\ M$$

Third, we deduce the $[C_2H_3O_2^-]$ from the chemical equilibrium equation.

$$HC_2H_3O_2 \rightleftharpoons H^+ + C_2H_3O_2^-$$

Notice that each H^+ is paired with a $C_2H_3O_2^-$ ion. This means

$$[H^+] = [C_2H_3O_2^-]$$

because each H^+ ion was originally associated with an acetate ($C_2H_3O_2^-$) ion. Therefore,

$$[C_2H_3O_2^-] = 3.2 \times 10^{-5}\ M \text{ also.}$$

Fourth we deduce the concentration of undissociated $HC_2H_3O_2$ from the initial concentration of $HC_2H_3O_2$ and the concentration of dissociated acetate ion ($C_2H_3O_2^-$). Since all of the acetic acid and acetate ion comes from the initial amount of acetic acid, the sum of the acetic acid and acetate concentrations must at all times equal the initial acetic acid concentration, i.e.,

$$[HC_2H_3O_2] + [C_2H_3O_2^-] = \text{Initial } HC_2H_3O_2 \text{ concentration}$$
$$= 9 \times 10^{-5}\ M$$

When this equation is solved for $[HC_2H_3O_2]$, we get

$$[HC_2H_3O_2] = 9 \times 10^{-5} - [C_2H_3O_2^-]$$

When the numerical value of $[C_2H_3O_2^-]$ is substituted into the equation, we find,

$$[HC_2H_3O_2] = (9 \times 10^{-5}) - (3.2 \times 10^{-5})$$

to find the difference on the right, we factor 10^{-5} from both terms.

$$\begin{aligned}[HC_2H_3O_2] &= 10^{-5}(9 - 3.2) \\ &= 10^{-5}(5.8) \\ &= 5.8 \times 10^{-5}\end{aligned}$$

We summarize the changes from the addition of $HC_2H_3O_2$ to the establishment of equilibrium in the table below.

		IONS		
		H^+	$C_2H_3O_2^-$	$HC_2H_3O_2$
Concentrations (M)	Initial	0*	0	9×10^{-5}
	Equilibrium	3.2×10^{-5}	3.2×10^{-5}	$9 \times 10^{-5} - [C_2H_3O_2^-]$ or equivalently 5.8×10^{-5}

Finally, the numerical values of the concentrations are substituted into the equation for the dissociation constant, and the value of K is calculated.

$$K = \frac{[H^+][C_2H_3O_2^-]}{[HC_2H_3O_2]}$$

$$= \frac{(3.2 \times 10^{-5})(3.2 \times 10^{-5})}{(5.8 \times 10^{-5})}$$

When the indicated operations are carried out and the result is rounded to two digits, it is

$$K = 1.8 \times 10^{-5}$$

14-2 Addition and Subtraction of Numbers in Scientific Notation

The solution of the preceding example required that the difference of two numbers in scientific notation be found. This will be a common occurrence in this sequence on applied quadratic equations. Therefore, we will focus some attention on this mathematical technique.

EXAMPLE:

Suppose

$$b^2 = 3.24 \times 10^{-10}$$

$$4ac = -.000072 \times 10^{-5}$$

Find $(b^2 - 4ac)$

*This is an approximation. There are actually H^+ ions present, due to the dissociation of H_2O into H^+ and OH^- ions, before the addition of $HC_2H_3O_2$, but an insignificant number in this instance.

SOLUTION:

Before the difference of the two numbers may be found, one of the numbers must be transformed so that both numbers make use of the same power of ten. It is simplest to transform

$$.000072 \times 10^{-5}$$

so that it is in the form

$$(\text{number}) \times 10^{-10}$$

where 10^{-10} matches the power of ten in the other number, 3.24×10^{-10}.

$$.000072 \times 10^{-5} = .000072 \times 10^{-10}$$

5 places

$$= 7.2 \times 10^{-10}$$

Finally, the difference may be found through an application of the distributive property.

$$b^2 - 4ac = (3.24 \times 10^{-10}) - (-7.2 \times 10^{-10})$$

$$= (3.24 \times 10^{-10}) + (7.2 \times 10^{-10})$$

When 10^{-10} is factored out,

$$b^2 - 4ac = 10^{-10}(3.24 + 7.2)$$

$$= 10^{-10}(10.44)$$

Or, in scientific notation,

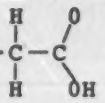
$$b^2 - 4ac = 10.44 \times 10^{-10}$$

One last point. The only new technique in this procedure is the final one, the use of the distributive property. Up to this step the procedure is identical to the procedure for comparing the sizes of two numbers in scientific notation (Section 12-6).

PROBLEM SET 14:

Calculate K for each of the following situations. Show calculations. Problems 1 through 3 refer to the dissociation of acetic acid.

$$HC_2H_3O_2 \rightleftharpoons H^+ + C_2H_3O_2^-$$

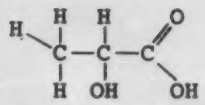
Acetic Acid = 

1. $[H^+] = 3 \times 10^{-5}$ M
   $[C_2H_3O_2^-] = 3 \times 10^{-5}$ M
   $[HC_2H_3O_2] = 5 \times 10^{-5}$ M
   K = ?

2. $[H^+] = 3 \times 10^{-4}$ M
   $[C_2H_3O_2^-] = 3 \times 10^{-4}$ M
   $[HC_2H_3O_2] = 5 \times 10^{-3}$ M

3. $[H^+] = 3 \times 10^{-3}$ M
   $[C_2H_3O_2^-] = 3 \times 10^{-3}$ M
   $[HC_2H_3O_2] = 5 \times 10^{-1}$ M

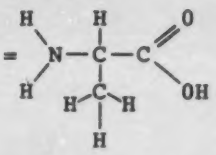
Problems 4 through 6 refer to lactic acid.

$HLac \rightleftharpoons H^+ + Lac^-$ Lactic Acid = 

$$K = \frac{[H^+][Lac^-]}{[HLac]}$$

4. $[H^+] = 1.4 \times 10^{-2}$ M
   $[Lac^-] = 1.4 \times 10^{-2}$ M
   $[HLac] = 1.4$ M
5. $[H^+] = 1.4 \times 10^{-3}$ M
   $[Lac^-] = 1.4 \times 10^{-3}$ M
   $[HLac] = 1.4 \times 10^{-2}$ M
6. $[H^+] = 1.4 \times 10^{-4}$ M
   $[Lac^-] = 1.4 \times 10^{-4}$ M
   $[HLac] = 1.4 \times 10^{-4}$ M

Problems 7 through 10 refer to the dissociation of the amino acid alanine in water.

$HAln \rightleftharpoons H^+ + Aln^-$ alanine = 

$$K = \frac{[H^+][Aln^-]}{[HAln]}$$

7. $[H^+] = 9 \times 10^{-2}$ M
   $[Aln^-] = 9 \times 10^{-2}$ M
   $[HAln] = 1.8$ M
8. $[H^+] = 3 \times 10^{-2}$ M
   $[Aln^-] = 3 \times 10^{-2}$ M
   $[HAln] = 2 \times 10^{-1}$ M
9. $[H^+] = 6 \times 10^{-3}$ M
   $[Aln^-] = 6 \times 10^{-3}$ M
   $[HAln] = 8 \times 10^{-3}$ M
10. $[H^+] = 1.2 \times 10^{-3}$ M
    $[Aln^-] = 1.2 \times 10^{-3}$ M
    $[HAln] = 3.2 \times 10^{-4}$ M

For Problems 11 through 20, perform the indicated additions and subtractions.

11. $(9 \times 10^{-4}) - (1.7 \times 10^{-3}) =$
12. $(42 \times 10^{-7}) - (.16 \times 10^{-5}) =$
13. $(.29 \times 10^{-10}) + (31 \times 10^{-13}) =$
14. $(.404 \times 10^{-3}) + (310 \times 10^{-5}) =$
15. $(16.4 \times 10^{-2}) + (.38 \times 10^{-1}) =$
16. $(210 \times 10^{-7}) - (.063 \times 10^{-4}) =$
17. $(.0003 \times 10^{-10}) + (700 \times 10^{-16}) =$
18. $(.047 \times 10^{-7}) - (93 \times 10^{-10}) =$
19. $(46 \times 10^{-8}) - (.467 \times 10^{-6}) =$
20. $(5731 \times 10^{-5}) - (5.726 \times 10^{-2}) =$

21. This problem calls for the construction of a graph which shows the relationship between the initial concentration of acetic acid (y) and the equilibrium $[H^+]$ (x). The graph may be used to answer questions such as, "What initial concentration of acetic acid is needed to make a solution with $[H^+] = 1 \times 10^{-4}$ M (pH = 4)?" The graph eliminates the need to perform any calculations.

To interpret a particular table entry, multiply it by (10^{-5}M). For example, find 2.2 in the x column. This 2.2 corresponds to a $[H^+]$ of 2.2×10^{-5} M.

a. Graph the data in the table.

x Equilibrium $[H^+]$ (10^{-5} M)	y Initial [Acetic Acid] (10^{-5} M)	pH
-3	2.0	
-2	0	
-1	-.5	
0	0	
1	1.5	
2.2	5	4.65
3.4	10	4.46
4.4	15	4.36
5.2	20	4.29
6.5	30	4.19
7.6	40	4.12
8.6	50	4.06
9.5	60	4.02
10.4	70	3.98
11.1	80	3.95
11.9	90	3.93
12.5	100	3.90

b. Draw the line $x = 1 \times 10^{-5}$ on the same graph. The portion of the graph to the right of this line describes the actual relationship between initial concentration and equilibrium $[H^+]$. The other points were included to demonstrate that the graph is a parabola.

c. From your graph find the equilibrium $[H^+]$ for an initial [Acetic Acid] of 65×10^{-5} M.

d. Find the initial [Acetic Acid] needed to achieve an equilibrium $[H^+] = 11.5 \times 10^{-5}$ M.

e. Find the initial [Acetic Acid] needed to achieve an equilibrium $[H^+] = 5 \times 10^{-5}$ M.

f. The $[H^+]$ of an acetic acid solution is found to be 4×10^{-5} M (pH = 4.4). What was the initial [Acetic Acid]?

g. The $[H^+]$ of an acetic acid solution is found to be 12.7×10^{-5} M (pH = 3.9). What was the initial [Acetic Acid]?

h. Suppose a biochemist needed to make an acetic acid solution with an equilibrium $[H^+] = 2 \times 10^{-5}$ M. What initial [Acetic Acid] would he prepare?

*22. For succinic acid $K = 6.4 \times 10^{-5}$.

a. On the same piece of graph paper as Problem 21 sketch in the following points for succinic acid.

X [Equilibrium] $[H^+]$ $(10^{-5}$ M)	Y Initial [Succinic Acid] $(10^{-5}$ M)
1	1.2
4.4	7.4
7.6	17
10.4	27
12.5	37

b. Sketch a line through the points you plotted in Part a.

c. For an equilibrium $[H^+] = 7.6 \times 10^{-5}$ M the initial [Acid] is (more, less) for succinic acid than for acetic.

d. For a given equilibrium $[H^+]$ (more, less) succinic acid is needed to achieve it than acetic.

e. Which is the stronger acid succinic or acetic?

f. For succinic acid $K = 6.4 \times 10^{-5}$. For acetic acid $K = 1.8 \times 10^{-5}$. Based on these two acids, the larger the value of K, the (stronger, weaker) the acid.

g. Since it always takes a (smaller, larger) initial [Succinic Acid] to achieve a particular pH, it may be concluded that succinic acid dissociates (more easily, less easily) than acetic acid.

23. A solution of nitrous acid in water is made up which has an initial concentration of 8×10^{-4} M.

When the pH is measured a short time later, it is found to be 3.4.

The chemical equation which describes equilibrium is

$$HNO_2 \rightleftharpoons H^+ + NO_2^-$$

The mathematical relationship between the different concentrations at equilibrium is given by the equation

$$K = \frac{[H^+][NO_2^-]}{[HNO_2]}$$

a. Use the pH given above and the table in Section 11 to calculate $[H^+]$.

b. The $[H^+]$ and the $[NO_2^-]$ must be (the same, different); therefore the $[NO_2^-]$ = ?

c. Since all the nitrous acid (HNO_2) and nitrite ion (NO_2^-) come from the initial amount of nitrous acid, the sum $[HNO_2] + [NO_2^-]$ at equilibrium must be equal to the (equilibrium, initial) concentration of HNO_2.

d. The equilibrium $[HNO_2]$ = ?

e. Calculate the dissociation constant for nitrous acid.

24. Norbert Numnitz drops a tablet of saccharin into his coffee to sweeten it up a little. (He doesn't use sugar because he has read the Nutrition Unit.) This produces an initial concentration of 2.75×10^{-3} M of saccharin.

He stirs his coffee absent mindedly for a couple of minutes and then checks the pH out of curiosity. It is 2.6.

In everything that follows assume that saccharin is an acid and all H^+ comes from the dissociation of saccharin. This is a fairly accurate assumption.

The dissociation of saccharin is described by

$$HSac \rightleftharpoons H^+ + Sac^-$$

and

$$K = \frac{[H^+][Sac^-]}{[HSac]}$$

a. Determine $[H^+]$

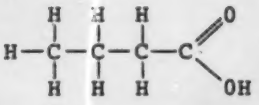
b. Determine $[Sac^-]$

c. Determine [HSac]

d. Calculate K

25. Butanoic acid forms when butter turns rancid. A solution of butanoic acid with an initial concentration of 1.76×10^{-3} M is made up.

Shortly an equilibrum is established which is described by

$HBut \rightleftharpoons H^+ + But^-$ and HBut = 

The pH of the solution is 3.8.

Calculate the dissociation constant.

26. A solution of vanillic acid is made with an initial concentration of 5.4×10^{-3} M. Vanillic acid dissociates in water as described by

$$HVan \rightleftharpoons H^+ + Van^-$$

The equilibrium pH was found to be 3.4.

Calculate the dissociation constant.

SECTION 15: CALCULATING THE INITIAL CONCENTRATION FOR A PARTICULAR $[H^+]$

15-1 An Example

It is often useful in the laboratory to be able to calculate the initial concentration needed to achieve a particular $[H^+]$. Dissociation constant equations enable this to be done.

PROBLEM:

A chemist in a brewery wants to make up an acetic acid solution with a $[H^+] = 10^{-4}$ M (pH = 4 to match the pH of beer). What must the initial concentration of $HC_2H_3O_2$ be to achieve an equilibrium $[H^+] = 10^{-4}$ M?

$$K = 1.8 \times 10^{-5}$$

SOLUTION:

We reason as follows. Since $[H^+] = 10^{-4}$ M, $[C_2H_3O_2^-]$ must also be 10^{-4} because the equilibrium produces H^+ and $C_2H_3O_2^-$ ions in equal numbers.

$$HC_2H_3O_2 \rightleftharpoons H^+ + C_2H_3O_2^-$$

Already we know three of the four values in the equation

$$K = \frac{[H^+][C_2H_3O_2^-]}{[HC_2H_3O_2]}$$

They are

$$K = 1.8 \times 10^{-5}$$

$$[H^+] = 10^{-4} \text{ M}$$

$$[C_2H_3O_2^-] = 10^{-4} \text{ M}$$

This means that we can solve for $[HC_2H_3O_2]$, the equilibrium concentration of undissociated acetic acid. We let

$$[HC_2H_3O_2] = z$$

because z is easier to write than $[HC_2H_3O_2]$, and substitute these values into the equation

$$K = \frac{[H^+][C_2H_3O_2^-]}{[HC_2H_3O_2]}$$

to obtain

$$1.8 \times 10^{-5} = \frac{(10^{-4})(10^{-4})}{z}$$

When both sides are multiplied by

$$\frac{z}{1.8 \times 10^{-5}}$$

we get

$$\cancel{1.8 \times 10^{-5}} \cdot \frac{z}{\cancel{1.8 \times 10^{-5}}} = \frac{(10^{-4})(10^{-4})}{\cancel{z}} \cdot \frac{\cancel{z}}{1.8 \times 10^{-5}}$$

$$z = \frac{(10^{-4})(10^{-4})}{1.8 \times 10^{-5}}$$

$$z = \frac{1}{1.8} \times 10^{-3}$$

$$z = .56 \times 10^{-3}$$

$$z = 5.6 \times 10^{-4}$$

We can now find the initial $[HC_2H_3O_2]$, our goal. Since all of the $HC_2H_3O_2$ and $C_2H_3O_2^-$ came from the original $HC_2H_3O_2$.

$$\text{initial } [HC_2H_3O_2] = \text{equilibrium } [C_2H_3O_2^-] + \text{equilibrium } [HC_2H_3O_2]$$

We now know both numbers on the right. When substituted,

$$\text{initial } [HC_2H_3O_2] = 1 \times 10^{-4} + 5.6 \times 10^{-4}$$

$$= 6.6 \times 10^{-4} \text{ M}$$

and we are finished. Our chemist friend would make up an acetic acid solution with an initial concentration of 6.6×10^{-4} M.

Since there are many different concentrations, chemicals, ions and so forth, it is a good idea to construct a table to keep track of them. The above problem is summarized in the table below.

		H^+	$C_2H_3O_2^-$	$HC_2H_3O_2$
Concentrations (M)	Initial	0	0	$10^{-4} + z$
	Equilibrium	10^{-4}	10^{-4}	z

It will often be our practice to construct the table first, before proceeding to the solution. This is similar to consulting a map before taking off for a particular destination.

15-2 Another Example

PROBLEM:

What initial concentration is required to achieve an acetic acid solution with a pH of 5 (an equilibrium $[H^+] = 10^{-5}$ M)?

SOLUTION:

We first use the idea that the equilibrium reaction

$$HC_2H_3O_2 \rightleftharpoons H^+ + C_2H_3O_2^-$$

requires that

$$[H^+] = [C_2H_3O_2^-]$$

Therefore

$$[C_2H_3O_2^-] = 10^{-5}\ M$$

at equilibrium also. Since we do not know the equilibrium $[HC_2H_3O_2]$, the concentration of undissociated acetic acid, we let

$$[HC_2H_3O_2] = z$$

Furthermore, we know that

$$10^{-5} + z = \text{Initial } [HC_2H_3O_2]$$

because all of the $C_2H_3O_2^-$ and $HC_2H_3O_2$ at equilibrium had to come from the initial $[HC_2H_3O_2]$.

We summarize this in a table,

		H^+	$C_2H_3O_2^-$	$HC_2H_3O_2$
Concentrations (M)	Initial	0	0	$10^{-5} + z$
	Equilibrium	10^{-5}	10^{-5}	z

and proceed to solve the problem.

$$K = \frac{[H^+][C_2H_3O_2^-]}{[HC_2H_3O_2]}$$

and

$$K = 1.8 \times 10^{-5}$$

We substitute the values in the table into the equation and solve for z.

$$1.8 \times 10^{-5} = \frac{(10^{-5})(10^{-5})}{z}$$

$$z = \frac{10^{-10}}{1.8 \times 10^{-5}}$$

$$\simeq .56 \times 10^{-5}$$

The initial $[HC_2H_3O_2]$ is therefore

$$1 \times 10^{-5} + x = 1 \times 10^{-5} + .56 \times 10^{-5}$$

$$= 1.56 \times 10^{-5}\ M$$

PROBLEM SET 15:

1. A chemist for a meat packing plant needs to make a nitrous acid solution with a pH = 3 to pass FDA regulations and please the owner. Nitrous acid dissociates in water as described by the equilibrium reaction

$$HNO_2 \rightleftharpoons H^+ + NO_2^-$$

$$K = \frac{[H^+][NO_2^-]}{[HNO_2]}$$

$$= 4 \times 10^{-4}$$

a. Copy the table and complete it.

		H^+	NO_2^-	HNO_2
Concentrations (M)	Initial	0	?	?
	Equilibrium	10^{-3}	?	z

b. Substitute the information in the table into the dissociation constant equation

$$4 \times 10^{-4} = \frac{[H^+][NO_2^-]}{[HNO_2]}$$

and solve for z, the equilibrium $[HNO_2]$.

c. Use the fact that all of the NO_2^- and HNO_2 that exists at equilibrium had to come from the initial HNO_2 to calculate the initial $[HNO_2]$.

2. A chemist needs to make a HNO_2 solution with a pH of 2 to study possible interaction of HNO_2 and the enzyme pepsin.

a. Construct a table similar to the one in Problem 1 that shows both the initial and equilibrium concentrations of H^+, NO_2^- and HNO_2.

b. Use the information in the table and the equation

$$4 \times 10^{-4} = \frac{[H^+][NO_2^-]}{[HNO_2]}$$

to calculate the equilibrium concentration of HNO_2.

c. Use the equilibrium $[NO_2^-]$ and $[HNO_2]$ to calculate the initial $[HNO_2]$.

3. A chemist needs to make up a HNO_2 solution with a pH of 2.3 ($[H^+] = 5 \times 10^{-3}$ M). What initial $[HNO_2]$ is required to achieve this equilibrium pH?

4. Malic acid is involved in the Krebs cycle. You may recall the Krebs cycle from the cell respiration game. A biochemist needs to make up a solution of malic acid which has an equilibrium pH of 1.4 ($[H^+] = 4 \times 10^{-2}$ M). $K = 4 \times 10^{-4}$.

$$H_2Mal \rightleftharpoons H^+ + HMal^-$$

What initial H_2Mal is required to achieve an equilibrium pH of 1.4?

5. Uric acid is an excretion product from the breakdown of nitrogen compounds. Suppose the biochemist in Problem 4 needs to make a uric acid (HUr) solution with an equilibrium

$$[H^+] = 2.6 \times 10^{-2}\ M$$

What initial [HUr] is required to achieve this $[H^+]$? ($K = 1.3 \times 10^{-4}$)

6. Tartaric acid (H_2Tar) is sometimes used as a preservative in cheese and other foods. ($K = 1 \times 10^{-3}$)

What initial $[H_2Tar]$ is required to achieve an equilibrium $[H^+] = 5 \times 10^{-4}$ M?

$$H_2Tar \rightleftharpoons H^+ + HTar^-$$

$$10^{-3} = \frac{[H^+][HTar^-]}{[H_2Tar]}$$

7. Benzoic acid may be used as a food preservative. $K = 6.3 \times 10^{-5}$

$$HBenz \rightleftharpoons H^+ + Benz^-$$

What initial [HBenz] is required to achieve an equilibrium pH of 4.2?

8. Butanoic acid is a fatty acid found in rancid butter and other foods.

$$K = 1.5 \times 10^{-5}$$

$$HBut \rightleftharpoons H^+ + But^-$$

What initial [HBut] is required to produce an equilibrium pH of 4.3?

SECTION 16: CHEMICAL EQUILIBRIA AND THE QUADRATIC FORMULA

In the previous section we found an initial concentration for a given $[H^+]$. This section deals with the reverse and will lead to a quadratic equation. The technique is useful for determining the pH of stock solutions when pH paper or a pH meter is not handy.

PROBLEM:

What is the equilibrium $[H^+]$ of a 10^{-5} M (initial) solution of acetic acid? $K = 1.8 \times 10^{-5}$.

SOLUTION:

Let $[H^+] = z$.

Since for every H^+ ion formed, a $C_2H_3O_2^-$ ion is formed,

$$[H^+] = [C_2H_3O_2^-]$$

or

$$[C_2H_3O_2^-] = z$$

Since the H^+ and $C_2H_3O_2^-$ ions came from the original $HC_2H_3O_2$, the concentration of $HC_2H_3O_2$ molecules in solution will be reduced by z. In other words, each $C_2H_3O_2^-$

ion in solution represents a former $HC_2H_3O_2$ molecule. So,

$$[HC_2H_3O_2] = 10^{-5} - z$$

at equilibrium. We summarize the changes in the table below.

		H^+	$C_2H_3O_2^-$ (Acetate)	$HC_2H_3O_2$ (Acetic Acid)
Concentration (M)	Initially	0	0	10^{-5}
	At Equilibrium	z	z	$10^{-5} - z$

Substitute the equivalent expressions for $[H^+]$, $[C_2H_3O_2^-]$, $[HC_2H_3O_2]$ and K into the equation

$$K = \frac{[H^+][C_2H_3O_2^-]}{HC_2H_3O_2}$$

to obtain

$$1.8 \times 10^{-5} = \frac{z \cdot z}{(10^{-5} - z)}$$

Multiply both sides by $(10^{-5} - z)$ to obtain

$$(1.8 \times 10^{-5})(10^{-5} - z) = z^2$$

An inspection of this equation reveals that it is a quadratic equation, although not in standard form. Since there is a z^2 on the right side, we can see that there will be a quadratic term, that is, one with a z^2 in it. There will be a linear term, one with a z in it and also a constant term. We proceed to put the equation into standard form.

The first step is to perform the indicated multiplication on the left side.

$$(1.8 \times 10^{-10}) - (1.8 \times 10^{-5})z = z^2$$

When z^2 is subtracted from both sides, we obtain

$$-z^2 + (1.8 \times 10^{-10}) - (1.8 \times 10^{-5})z = 0$$

After multiplying both sides by -1 and rearranging, the equation becomes

$$z^2 + (1.8 \times 10^{-5})z - (1.8 \times 10^{-10}) = 0$$

An inspection of this equation shows that the quadratic coefficients a, b, and c are

$$a = 1 \quad b = 1.8 \times 10^{-5} \quad c = -1.8 \times 10^{-10}$$

This is as far as we will go in this section with the solution of problems of this type. We will concentrate on setting up the quadratic equation and converting it to standard form. We will leave the solution for the next section. The postponement of the actual solution is part of an attempt to simplify a long and complicated pattern of solution.

PROBLEM SET 16:

1. Citric acid is one of the compounds formed in the Krebs cycle.

A solution of citric acid in water is made up that has an initial concentration of 10^{-3} M. Citric acid dissociates as described below.

$$H_3Cit \rightleftharpoons H^+ + H_2Cit^-$$

a. Copy the table below and complete it.

		H^+	H_2Cit^-	H_3Cit
Concentration (M)	Initially	?	0	10^{-3}
	Equilibrium	z	?	?

b. At equilibrium

$$9 \times 10^{-4} = \frac{[H^+][H_2Cit^-]}{[H_3Cit]}$$

Substitute the literal expressions for the equilibrium concentrations into this equation.

c. Transform the equation of Part b into a quadratic equation in standard form.

d. a = ?, b = ?, c = ?

2. A solution of citric acid is made up which has an initial concentration of 5×10^{-3} M.

a. Construct a table similar to the one in Problem 1 that shows both the initial and equilibrium concentrations of H^+, H_2Cit^- and H_3Cit.

b. At equilibrium

$$9 \times 10^{-4} = \frac{[H^+][H_2Cit^-]}{[H_3Cit]}$$

Substitute the literal expressions for the different equilibrium concentrations into this equation.

c. Transform the equation of Part b into a quadratic equation in standard form.

d. a = ?, b = ? c = ?

3. A citric acid solution is made up with an initial concentration of 10^{-2} M.

a. Construct a table similar to the one in Problem 1 that shows both the initial and equilibrium concentrations of H^+, H_2Cit^- and H_3Cit.

b. Use the dissociation constant equation to develop a quadratic equation in standard form which describes the equilibrium relations.

c. a = ?, b = ?, c = ?

In Problems 4 through 10 determine the quadratic coefficients a, b and c.

4. An ascorbic acid (better known as "vitamin C") solution is made up with an initial concentration of 1×10^{-4} M.

$$K = 8 \times 10^{-5}$$

$$H_2Asc \rightleftharpoons H^+ + HAsc^-$$

5. A pyruvic acid (of glycolysis fame) solution is made up which has an initial concentration of 1×10^{-2} M.

$$K = 3.2 \times 10^{-3}$$

$$HPyr \rightleftharpoons H^+ + Pyr^-$$

6. A succinic acid (memorable for its part in the long-running metabolic melodrama, "The Krebs Cycle") solution is made up with an initial concentration of 10^{-4} M.

$$K = 6.4 \times 10^{-5}$$

$$H_2Suc \rightleftharpoons H^+ + HSuc^-$$

7. Malic acid (a co-star in "The Krebs Cycle") is added to water in an amount sufficient to make a solution with an initial concentration of 10^{-4} M.

$$K = 4 \times 10^{-4}$$

$$H_2Mal \rightleftharpoons H^+ + HMal^-$$

8. Glycine (a member of that chain gang known as "The Amino Acids") is added to water making a solution with an initial concentration of 5×10^{-3} M.

$$K = 4.6 \times 10^{-3}$$

$$HGly \rightleftharpoons H^+ + Gly^-$$

9. Enough saccharin (an acidic substitute for sweet sucrose) is added to water to make a solution with an initial concentration of 2.5×10^{-2} M.

$$K = 2.5 \times 10^{-2}$$

$$HSac \rightleftharpoons H^+ + Sac^-$$

10. Lactic acid (the culprit responsible for tired muscles immediately after hard exercise) is added to water in an amount sufficient to make an initial concentration of 3×10^{-2} M.

$$K = 1.4 \times 10^{-4}$$

$$HLac \rightleftharpoons H^+ + Lac^-$$

SECTION 17: THE SOLUTION OF QUADRATIC EQUATIONS RELATING TO CHEMICAL EQUILIBRIA

In this section we will complete the solution of the problem which was begun in Section 16. We repeat the statement of the problem.

PROBLEM:

What is the equilibrium $[H^+]$ of a 10^{-5} M (initial concentration) solution of acetic acid? $K = 1.8 \times 10^{-5}$.

SOLUTION:

In Section 16 we let

$$z = [H^+]$$

and derived the quadratic equation

$$z^2 + (1.8 \times 10^{-5})z - (1.8 \times 10^{-10}) = 0$$

We will now use the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to solve for z, the $[H^+]$. In the quadratic equation above, the numerical values of a, b and c are

$$a = 1 \quad b = 1.8 \times 10^{-5} \quad c = -1.8 \times 10^{-10}$$

When these numerical values are substituted into the quadratic formula, then (brace yourself)

$$z = \frac{-(1.8 \times 10^{-5}) \pm \sqrt{(1.8 \times 10^{-5})^2 - 4(-1.8 \times 10^{-10})}}{2}$$

Now we concentrate on the radical and temporarily ignore the rest of the expression.

First we compute $(1.8 \times 10^{-5})^2$.

$$(1.8 \times 10^{-5})^2 = 3.24 \times 10^{-10}$$

Second we calculate $-4(-1.8 \times 10^{-10})$.

$$-4(-1.8 \times 10^{-10}) = 7.2 \times 10^{-10}$$

Therefore,

$$\sqrt{(1.8 \times 10^{-5})^2 - 4(-1.8 \times 10^{-10})} = \sqrt{(3.24 \times 10^{-10}) + (7.2 \times 10^{-10})}$$

Third, we compute the sum of the two numbers under the radical sign.

$$= \sqrt{10^{-10}(3.24 + 7.2)}$$

$$= \sqrt{10^{-10}(10.44)}$$

$$= \sqrt{10.44 \times 10^{-10}}$$

Fourth, we determine the square root. Since the power of ten is already even, we do not have to worry about this detail. The square root may be found by referring to the square root table at the end of this unit.

$$\approx 3.23 \times 10^{-5}$$

We can now return to the original equation and substitute 3.23×10^{-5} for the radical expression.

$$z \approx \frac{-(1.8 \times 10^{-5}) \pm (3.23 \times 10^{-5})}{2}$$

After performing the indicated addition and subtraction

$$z \approx \frac{1.43 \times 10^{-5}}{2}$$

$$z \approx \frac{-5.03 \times 10^{-5}}{2}$$

When the division by 2 is performed we obtain the two values for z.

$$z \approx .715 \times 10^{-5}$$

$$z \approx -2.515 \times 10^{-5}$$

One solution is negative. But z is supposed to be $[H^+]$--we don't have any way of making sense of a negative concentration, and now is not the time to invent one. We conclude that the negative solution has no meaning in our particular example, so we will ignore it.

That leaves us with the first solution,

$$z \approx .715 \times 10^{-5}$$

which we round back to two digits because we don't want anyone to think that we're implying three-digit accuracy.

$$z \approx .72 \times 10^{-5}$$

or, equivalently,

$$z \approx 7.2 \times 10^{-6}$$

We conclude that

$$[H^+] \approx 7.2 \times 10^{-6} \text{ M}$$

And, consequently,

$$[C_2H_3O_2^-] \approx 7.2 \times 10^{-6} \text{ M}$$

Finally, the concentration of undissociated acetic acid may be found. Recall that the initial $[HC_2H_3O_2] = 10^{-5}$ M. We rewrite this number in an equivalent form (to make it easier to subtract 7.2×10^{-6} from it).

$$[HC_2H_3O_2] = 10 \times 10^{-6} \text{ M}$$

Since $([HC_2H_3O_2] + [C_2H_3O_2^-])$ must always be equal to this initial concentration, we can substitute the numerical value of $[C_2H_3O_2^-]$ and solve for $[HC_2H_3O_2]$.

$$[HC_2H_3O_2] + [C_2H_3O_2^-] = 10 \times 10^{-6} \text{ M}$$

$$[HC_2H_3O_2] + 7.2 \times 10^{-6} = 10 \times 10^{-6} \text{ M}$$

Therefore

$$[HC_2H_3O_2] = 2.8 \times 10^{-6} \text{ M}$$

This means that most of the acetic acid is dissociated at an initial concentration of 10^{-5} M.

PROBLEM SET 17:

For each of the following problems use these rules to state the number of digits in your answer.

(1) z: Round your result to two digits.

(2) Initial Concentration -z: Use as many digits as necessary to express the difference.

EXAMPLE:

$z = 5.4 \times 10^{-7}$

$[H^+] = 5.4 \times 10^{-7}$ M

$[\text{Anion}^-] = 5.4 \times 10^{-7}$ M

$$\text{Initial Concentration} - z = (1 \times 10^{-4}) - (5.4 \times 10^{-7}) \text{ M}$$
$$= (1000 \times 10^{-7}) - (5.4 \times 10^{-7}) \text{ M}$$
$$= 994.6 \times 10^{-7} \text{ M}$$
$$= 9.946 \times 10^{-5} \text{ M}$$

1. Nitrous acid may be used as a food preservative. In aqueous solution the following chemical equilibrium is established.

$$HNO_2 \rightleftharpoons H^+ + NO_2^-$$

Nitrous Acid — Hydrogen Ion — Nitrite Ion

It has been demonstrated that HNO_2 is the active ingredient which counteracts the growth of botulism bacteria. It is worthwhile, therefore, to be able to calculate the $[HNO_2]$ of a solution. The equilibrium concentrations are governed by the equation

$$K = \frac{[H^+][NO_2^-]}{[HNO_2]}$$

and K has been determined to be 4.6×10^{-4}. Assume that the initial concentration of HNO_2 (nitrous acid) is 10^{-3} M.

a. Let the concentration of NO_2^- (nitrite ion) at equilibrium be represented by z. Use

$$HNO_2 \rightleftharpoons H^+ + NO_2^-$$

to state why $[H^+]$ should also be represented by z.

b. State why $[HNO_2]$ at equilibrium should be represented by the expression $(10^{-3} - z)$.

c. Copy the table below and complete it.

		H^+	NO_2^-	HNO_2
Concentration (M)	Initial	?	?	?
	Equilibrium	z	z	$10^{-3} - z$

d. Substitute the expressions in the table into the dissociation constant equation.

$$K = \frac{[H^+][NO_2^-]}{[HNO_2]}$$

e. Transform the equation in Part d into a quadratic equation in standard form.

f. a = ?, b = ?, c = ?

g. Find the roots of the equation.

h. Why is only the positive root of interest?

i. What are the equilibrium concentrations of H^+, NO_2^-, and HNO_2?

2. Tartaric acid is added to cheese products, candy and bakery products. It is used to adjust the acidity of the food to a desired level. Possible side-effects of the dissociation of H_2Tar will be examined later. The dissociation is described by the equilibrium reaction

$$H_2Tar \rightleftharpoons H^+ + HTar^-$$

The dissociation constant for this reaction is $K = 1 \times 10^{-3}$. The concentrations at equilibrium are governed by the equation

$$K = \frac{[H^+][HTar^-]}{[H_2Tar]}$$

a. Let $z = [HTar^-]$. Let initial $[H_2Tar] = 2 \times 10^{-2}$ M. Construct a table similar to the one in Problem 1 which shows the initial and equilibrium concentrations of H^+, $HTar^-$ and H_2Tar.

b. Use the information in the table and the dissociation constant equation to derive a quadratic equation in standard form.

c. Solve the quadratic equation. Ignore the negative solution.

d. What are the equilibrium concentrations of H^+, $HTar^-$ and H_2Tar?

3. This problem deals once again with tartaric acid. This time we let

$$\text{initial } [H_2Tar] = .2 \text{ M}$$

a more common concentration in foods. $K = 1 \times 10^{-3}$.

a. Construct a table similar to the one in Problem 1 which shows the initial and equilibrium concentrations of H^+, $HTar^-$ and H_2Tar.

b. Use the information in the table and the dissociation constant equation to derive a quadratic equation in standard form.

c. Solve the quadratic equation. Ignore the negative solution.

d. What are the equilibrium concentrations of H^+, $HTar^-$ and H_2Tar?

4. Determine the equilibrium concentrations of H^+, $HAsc^-$ and H_2Asc for a 1×10^{-4} M (initial) concentration of ascorbic acid (H_2Asc).

Use a, b and c derived in Problem 4, Problem Set 16. $K = 8 \times 10^{-5}$

5. Determine the equilibrium concentrations of H^+, Pyr^- and HPyr for a 1×10^{-2} M (initial concentration) solution of pyruvic acid (HPyr).

Use a, b and c derived in Problem 5, Problem Set 16. $K = 3.2 \times 10^{-3}$

6. Determine the equilibrium concentrations of H^+, $HSuc^-$ and H_2Suc for a 10^{-4} M (initial) concentration of succinic acid (H_2Suc).

Use a, b and c derived in Problem 6, Problem Set 16. $K = 6.4 \times 10^{-5}$

7. Determine the equilibrium concentrations of H^+, $HMal^-$ and H_2Mal for a 10^{-4} M (initial) solution of malic acid (H_2Mal).

Use a, b and c as determined in Problem 7, Problem Set 16. $K = 4 \times 10^{-4}$

8. Determine the equilibrium concentrations of H^+, Gly^- and HGly for a 5×10^{-3} M (initial) concentration of glycine (HGly).

Use a, b and c derived in Problem 8, Problem Set 16. $K = 4.6 \times 10^{-3}$

9. Determine the equilibrium concentrations of H^+, Sac^- and HSac for a 2.5×10^{-2} M (initial) solution of saccharin (HSac).

Use a, b and c derived in Problem 9, Problem Set 16. $K = 2.5 \times 10^{-2}$

10. Determine the equilibrium concentrations of H^+, Lac^- and HLac for a 3×10^{-2} M (initial) concentration of lactic acid (HLac).

Use a, b and c derived in Problem 10, Problem Set 16. $K = 1.4 \times 10^{-4}$

11. Carbonated Beverages, Part 1 (Part 2 in Problem Set 18)

Carbon dioxide is added to water under high pressure until the concentration of CO_2 in the water is about .15 M. Some of the dissolved CO_2 combines with water and then dissociates into H^+ and HCO_3^- (bicarbonate) as described by the chemical equilibrium

$$CO_2 + H_2O \rightleftharpoons H^+ + HCO_3^-$$

The concentrations at equilibrium are governed by the equation

$$4 \times 10^{-7} = \frac{[H^+][HCO_3^-]}{[CO_2]}$$

a. Construct a table which shows the initial and equilibrium concentrations of H^+, HCO_3^- and CO_2. Neglect $[H_2O]$. Its concentration changes so little that it is common practice to ignore it in equilibrium calculations.

b. Use the information in the table to derive a quadratic equation in standard form.

c. Evaluate b^2.

d. Evaluate -4ac.

e. Which number, (b^2) or (-4ac), is more than a million times (10^6) larger than the other?

f. Ignore the smaller number. Evaluate $\sqrt{b^2 - 4ac}$.

g. What are the equilibrium concentrations of H^+, HCO_3^- and CO_2?

SECTION 18: DISSOCIATION CONSTANTS AND THE BIOCHEMICAL BREW

So far, we have studied only solutions formed by adding chemicals to pure water. This is a situation rarely encountered in the human body. Typically, there will be many other ions and chemicals dissolved in any biochemical brew, such as blood, stomach juice or urine.

The example below illustrates how to handle this sort of situation. Problem 11 in the problem set is a follow-up to this example. These two problems investigate whether potassium and tartrate will precipitate in the intestinal tract.

Chemists have discovered that if KHTar precipitates, each molecule going out of solution will take two molecules of water with it. This tends to dry out the feces, causing constipation.

PROBLEM:

Enough tartaric acid is added to stomach juice to give an initial concentration of .02 M, as a consequence of eating about 2 oz of cheese. The concentration of H^+ ions in the stomach juice was already 1×10^{-2} M (pH of 2) before eating the cheese, mainly due to the HCl produced in the stomach.

The reaction is

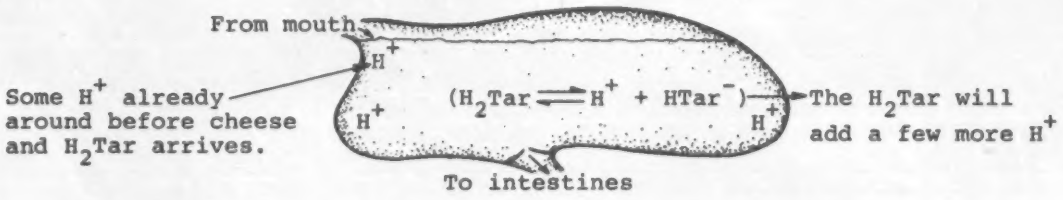
$$H_2Tar \rightleftharpoons H^+ + HTar^-$$

Notice that H_2Tar is not a precipitate. It is water-soluble, like sucrose. The dissociation equation is

$$K = \frac{[H^+][HTar^-]}{[H_2Tar]}$$

where $K = 1 \times 10^{-3}$.

A diagram may be of some help in sorting out the characters in this biochemical drama.


From mouth
Some H+ already around before cheese and H2Tar arrives.
(H2Tar ⇌ H+ + HTar−)
The H2Tar will add a few more H+
To intestines

What will be the equilibrium concentrations of H^+, $HTar^-$ and H_2Tar?

SOLUTION:

First we write an expression for the new equilibrium $[H^+]$. Since H_2Tar dissociates to form H^+ ions as described by the equilibrium reaction

$$H_2Tar \rightleftharpoons H^+ + HTar^-$$

some additional H^+ will be contributed by the H_2Tar added. We let

$$z = \text{additional } [H^+] \text{ contributed by } H_2Tar$$

Since the $[H^+]$ was already 1×10^{-2} M before the H_2Tar arrived, the total $[H^+]$ will be 1×10^{-2} M + z M, i.e.,

$$\underset{\text{Old } [H^+]}{(1 \times 10^{-2})} + \underset{\substack{\uparrow \\ \text{Additional} \\ [H^+] \text{ from } H_2Tar}}{z} = \text{New Equilibrium } [H^+]$$

Second, we write an expression for the equilibrium $[HTar^-]$. Since each new H^+ produced also produces an $HTar^-$ ion, the $[HTar^-]$ at equilibrium will be identical to the increase in $[H^+]$.

$$z = \text{Equilibrium } [HTar^-]$$

Third, we write an expression for the equilibrium $[H_2Tar]$, the concentration of undissociated tartaric acid. We reason exactly as we have in earlier lessons. The sum of the $HTar^-$ and H_2Tar concentrations must at all times equal the initial amount. This is because $HTar^-$ and H_2Tar can come only from the initial amount.

$$[HTar^-] + [H_2Tar] = \underset{\substack{\uparrow \\ \text{Initial} \\ [H_2Tar]}}{2 \times 10^{-2}} \text{ M}$$

Since $[HTar^-] = z$,

$$z + [H_2Tar] = 2 \times 10^{-2}$$

Or, equivalently,

$$[H_2Tar] = (2 \times 10^{-2}) - z$$

Finally we put all of this information in a table where it may be easily seen.

		H^+	$HTar^-$	H_2Tar
Concentrations (M)	Initial	1×10^{-2}	0	2×10^{-2}
	Equilibrium	$(1 \times 10^{-2}) + z$	z	$(2 \times 10^{-2}) - z$

This table allows us to focus on the new feature that is being introduced in this section. Previously, the initial $[H^+]$ was zero, now it is not.

Proceeding with the solution, we substitute the information in the table into the dissociation constant equation.

$$K = \frac{[H^+][HTar^-]}{[H_2Tar]}$$

$$1 \times 10^{-3} = \frac{\{(1 \times 10^{-2}) + z\}(z)}{(2 \times 10^{-2}) - z}$$

Multiply both sides by $(2 \times 10^{-2}) - z$ to obtain

$$(1 \times 10^{-3})\{(2 \times 10^{-2}) - z\} = \{(1 \times 10^{-2}) + z\}(z)$$

Carry out the implied multiplications to get

$$(2 \times 10^{-5}) - (1 \times 10^{-3})z = (1 \times 10^{-2})z + z^2$$

Subtract the left side from both sides and rearrange.

$$0 = z^2 + (1 \times 10^{-2})z + (1 \times 10^{-3})z - (2 \times 10^{-5})$$

$$0 = z^2 + (1.1 \times 10^{-2})z - (2 \times 10^{-5})$$

Now apply the quadratic formula.

$$z = \frac{-(1.1 \times 10^{-2}) \pm \sqrt{(1.1 \times 10^{-2})^2 - 4(-2 \times 10^{-5})}}{2}$$

Temporarily ignore the rest of the expression, and concentrate on simplifying the radical.

$$b^2 = (1.1 \times 10^{-2})^2 \qquad -4ac = -4(-2 \times 10^{-5})$$

$$= (1.1)^2 \times (10^{-2})^2 \qquad = 8 \times 10^{-5}$$

$$= 1.21 \times 10^{-4}$$

$$b^2 - 4ac = (1.21 \times 10^{-4}) + (8 \times 10^{-5})$$

Rewrite 8×10^{-5} so that it uses 10^{-4}. Then

$$b^2 - 4ac = (1.21 \times 10^{-4}) + (.8 \times 10^{-4})$$

$$= 2.01 \times 10^{-4}$$

Take the square root

$$\sqrt{b^2 - 4ac} \approx 1.418 \times 10^{-2}$$

$$\approx 1.42 \times 10^{-2}$$

Substitute this number back into the complete expression for z.

$$z \approx \frac{-(1.1 \times 10^{-2}) \pm (1.42 \times 10^{-2})}{2}$$

One of these solutions is negative. We will ignore it, due to our irrational prejudice against negative concentrations. That leaves us with the other solution.

$$z \approx \frac{-(1.1 \times 10^{-2}) + (1.42 \times 10^{-2})}{2}$$

$$\approx \frac{.32 \times 10^{-2}}{2}$$

$$\approx .16 \times 10^{-2}$$

Now that we know the numerical value of z, we can find the numerical values of all the equilibrium concentrations.

$[H^+] = (1 \times 10^{-2}) + z$

$\approx (1 \times 10^{-2}) + (.16 \times 10^{-2})$

$\approx 1.16 \times 10^{-2}$ M

$[HTar^-] = z$

$\approx .16 \times 10^{-2}$ M

$\approx 1.6 \times 10^{-3}$ M

$[H_2Tar] = (2 \times 10^{-2}) - z$

$\approx (2 \times 10^{-2}) - (.16 \times 10^{-2})$

$\approx 1.84 \times 10^{-2}$ M

PROBLEM SET 18:

1. A grapefruit is about 1.3% citric acid. Eating half a grapefruit on an empty stomach results in an initial concentration of citric acid of about 5×10^{-2} M. The pH in the stomach ranges from 1 to 3. We will find the change in pH due to eating half a grapefruit.

First we assume a $[H^+] = 1.3 \times 10^{-3}$ M for the stomach before ingestion of the grapefruit (pH ≈ 2.9).

a. Let the initial concentrations be

$$H_3Cit \rightleftharpoons H^+ + H_2Cit^-$$

$$[H^+] = 1.3 \times 10^{-3} \text{ M}$$

$$[H_2Cit^-] = 0$$

$$[H_3Cit] = 5 \times 10^{-2} \text{ M}$$

Let z = the increase in $[H^+]$ due to $H_3Cit \rightleftharpoons H^+ + H_2Cit^-$

Construct a table which shows the initial and equilibrium concentrations of H^+, H_2Cit^- and H_3Cit.

b. Substitute the expressions for the equilibrium concentrations into the dissociation constant equation

$$7 \times 10^{-4} = \frac{[H^+][H_2Cit^-]}{[H_3Cit]}$$

c. Transform the equation of Part b into a quadratic equation in standard form. Show your work. The result should be

$$0 = z^2 + z(2 \times 10^{-3}) - (3.5 \times 10^{-5})$$

d. Solve the equation of Part c for z. Show your work. The result should be

$$z = 5 \times 10^{-3}$$

e. What are the equilibrium values of $[H^+]$, $[H_2Cit^-]$, and $[H_3Cit]$?

f. What is the equilibrium pH? How much did it change?

2. Next we assume an initial $[H^+] = .93 \times 10^{-2}$ M (pH ≈ 2) for the $[H^+]$ before the arrival of the grapefruit.

a. Initially the concentrations are

$$[H^+] = .93 \times 10^{-2} \text{ M}$$

$$[H_2Cit^-] = 0$$

$$[H_3Cit] = 5 \times 10^{-2} \text{ M}$$

After the equilibrium described by the chemical reactions

$$H_3Cit \rightleftharpoons H^+ + H_2Cit^-$$

is established, there will be a $[H_2Cit^-]$ of z, a $[H^+] = (.93 \times 10^{-2}) + z$ and a $[H_3Cit] = (5 \times 10^{-2}) - z$.

Construct a table showing the initial and equilibrium concentrations.

b. Substitute the expressions for the equilibrium concentrations into the dissociation constant equation

$$7 \times 10^{-4} = \frac{[H^+][H_2Cit^-]}{[H_3Cit]}$$

c. Transform the equation of Part b into a quadratic equation in standard form. Show your work. The final equation will be

$$0 = z^2 + z(1 \times 10^{-2}) - (3.5 \times 10^{-5})$$

d. Substitute the quadratic coefficients of Part c into the quadratic formula and solve for z. Show your work. The result should be

$$z = 2.745 \times 10^{-3}$$

e. Substitute the numerical value for z (Part d) into the expressions for the equilibrium concentrations to find $[H^+]$, $[H_2Cit^-]$ and $[H_3Cit]$.

f. What is the equilibrium pH? How much did it change from the initial pH of 2?

3. Ascorbic acid is added to a solution which is already acidic. Initially

$$[H^+] = 1 \times 10^{-4} \text{ M}$$

$$[H_2Asc] = 5 \times 10^{-5} \text{ M}$$

K for ascorbic acid is 8×10^{-5} and

$$8 \times 10^{-5} = \frac{[H^+][HAsc^-]}{[H_2Asc]}$$

at equilibrium.

Find the equilibrium concentrations of H^+, $HAsc^-$ and H_2Asc.

4. Glycine is added to a solution which is already acidic. Initially

$$[H^+] = 1 \times 10^{-3} \text{ M}$$

$$[HGly] = 8 \times 10^{-3} \text{ M}$$

K for glycine is 5×10^{-3}.

Find the equilibrium concentrations of H^+, Gly^- and HGly.

5. Pyruvic acid is added to a solution which is already acidic. Initially

$$[H^+] = 8 \times 10^{-4} \text{ M}$$

$$[HPyr] = 3.75 \times 10^{-3} \text{ M}$$

K for pyruvic acid is 3.2×10^{-3}.

Find the equilibrium concentrations of H^+, Pyr^- and HPyr.

6. Malic acid is added to a solution which is already acidic. Initially

$$[H^+] = 2 \times 10^{-4} \text{ M}$$

$$[H_2Mal] = 4 \times 10^{-4} \text{ M}$$

K for malic acid is 4×10^{-4}.

Find the equilibrium concentrations of H^+, $HMal^-$ and H_2Mal.

7. Nitrous acid is added to a solution which is already acidic. Initially

$$[H^+] = 9 \times 10^{-4} \text{ M}$$

$$[HNO_2] = 3.5 \times 10^{-4} \text{ M}$$

K for nitrous acid is 4×10^{-4}.

Find the equilibrium $[H^+]$, $[NO_2^-]$ and $[HNO_2]$.

8. Saccharin is added to a solution which is already acidic. Initially

$$[H^+] = 1.5 \times 10^{-2} \text{ M}$$

$$[HSac] = 6.5 \times 10^{-2} \text{ M}$$

K for saccharin is 2.5×10^{-2}. Find the equilibrium $[H^+]$, $[Sac^-]$ and [HSac].

9. Tartaric acid is added to a solution which is already acidic. Initially

$$[H^+] = 9 \times 10^{-3} \text{ M}$$

$$[H_2Tar] = 6 \times 10^{-1} \text{ M}$$

K for tartaric acid is 1×10^{-3}. Find the equilibrium $[H^+]$, $[HTar^-]$ and $[H_2Tar]$.

10. Carbonated Beverages, Part 2

Recall that in soft drinks, CO_2 is pumped into water until a $[H^+]$ of roughly 2.5×10^{-4} M is achieved. In order to increase the tartness of a carbonated drink, bottlers typically add more acid. Citric, tartaric and malic acids are added to fruit drinks. They are natural acids derived from fruits. Phosphoric acid is the choice for cola drinks. Enough phosphoric acid is added to achieve an initial phosphoric acid concentration of 8×10^{-3} M. This works out to be a little less than the 1% maximum found in soft drinks. The table shows the expressions for the initial and equilibrium concentrations for this situation.

		H^+	$H_2PO_4^-$	H_3PO_4
Concentrations (M)	Initial	$.25 \times 10^{-3}$	0	8×10^{-3}
	Equilibrium	$(.25 \times 10^{-3}) + z$	z	$(8 \times 10^{-3}) - z$

The dissociation equation for the following reaction

$$H_3PO_4 \rightleftharpoons H^+ + H_2PO_4^-$$

is

$$K = \frac{[H^+][H_2PO_4^-]}{[H_3PO_4]}$$

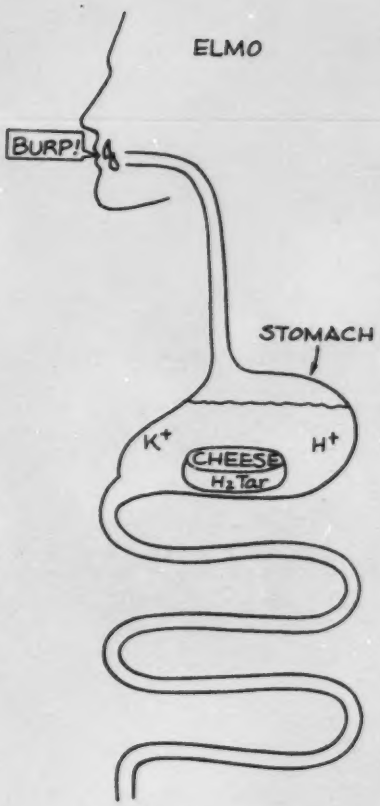
and $K = 8 \times 10^{-3}$

a. Determine z. Round z to two digits.

b. Determine the equilibrium $[H^+]$, $[H_2PO_4^-]$ and $[H_3PO_4]$.

c. What is pH?

11. Potassium forms a slightly soluble compound with $HTar^-$ ions (KHTar). $HTar^-$ ions are formed when tartaric acid is added to stomach juice. Tartaric acid is an additive in processed cheese. Potassium is present in stomach juice. An interesting question to explore is, "Does a precipitate of KHTar form when cheese is in the stomach?" The whole confusing cast of chemical characters is shown in the diagram.



a. An $HTar^-$ concentration of about 1.6×10^{-3} M may be obtained by eating 2 oz of processed cheese, approximately the amount of cheese in one grilled cheese sandwich. This calculation is shown in the sample problem in Section 18. It is based on the assumption of 600 ml of solution in the stomach and an initial $[H^+]$ of 1×10^{-2} M.

A typical $[K^+]$ in the stomach is 2×10^{-2} M. K_{sp} for KHTar is about 4×10^{-4}. Therefore, initially

$$[K^+] = 2 \times 10^{-2} \text{ M}$$

$$[HTar^-] = 1.6 \times 10^{-3} \text{ M}$$

Calculate $[K^+][HTar^-]$ and answer the question, "Does KHTar precipitate?" Recall that if

$$[K^+][HTar^-] > K_{sp} \quad (K_{sp} = 4 \times 10^{-4})$$

then a precipitate forms.

1. Does $KHTar \cdot 2H_2O$ precipitate?
2. Does Elmo get constipated?
3. If so, is it psychosomatic?
4. Does anybody care?

b. Now we will see if KHTar precipitates in Elmo's colon. If it does, it will tend to dry out the feces because each KHTar molecule going out of solution will carry two water molecules with it to form the hydrated molecule $KHTar \cdot 2H_2O$. Before this may be found out we must determine the equilibrium $[HTar^-]$ and $[K^+]$ in the colon. Then it may be determined whether

$$[K^+][HTar^-] > K_{sp}$$

But before the equilibrium $[HTar^-]$ can be calculated, we must first deduce the initial $[H_2Tar]$.

By the time the contents of the stomach reach the colon, 90% of the water has been removed. This will increase the initial $[H_2Tar]$ from .02 M to .2 M. Why?

c. To find the equilibrium $[HTar^-]$ in the colon, we must know something about the $[H^+]$. Since the body adjusts the $[H^+]$ upstream in the duodenum, the situation is unlike others we have analyzed in this section. We'll have to use a different approach. Laboratory analyses have shown that the pH of feces is in the range 4.6 to 8.4. We will assume that the contents of Elmo's colon has the maximum $[H^+]$. pH = 4.6 means $[H^+] = 2.5 \times 10^{-5}$ M.

Copy and complete the table. Notice that there is no "+z" term in the equilibrium $[H^+]$ expression.

		IONS		
		H^+	$HTar^-$	H_2Tar
Concentration (M)	Initial		0	.2
	Equilibrium	2.5×10^{-5}	z	?

You will need to use the idea that $[HTar^-] + [H_2Tar] = .2$ M <u>at all times</u>.

d. Substitute the expressions for the equilibrium concentrations into the dissociation constant equation and solve for z. Hint: The equation will <u>not</u> be a quadratic.

$$1 \times 10^{-3} = \frac{[H^+][HTar^-]}{[H_2Tar]}$$

e. A typical $[K^+]$ concentration in the colon is 10^{-1} M. So, in the colon

$$[K^+] = 10^{-1} \text{ M}$$

$$[HTar^-] = \text{answer to Part d}$$

Calculate $[K^+][HTar^-]$ and answer the question, "Does KHTar precipitate?"

f. Now suppose that the $[H^+]$ in Elmo's colon was less than 2.5×10^{-5} M. It can be shown that this will lead to an even larger equilibrium $[HTar^-]$.

Will $[K^+][HTar^-] > K_{sp}$ for larger values of $[HTar^-]$?

g. Will Elmo get constipated?

SECTION 19: REVIEW

The problems which follow will not all be of the same type. Some will be like those in Section 16, some like those in Section 18 and so forth. There are a couple of ways to deal with this complication. One is to thumb through the preceding sections until you find a problem of the type you are trying to solve. Another, better alternative is to approach each problem in a way that will lead to the correct pattern of solution. This is a better method because it will solve many kinds of problems that we have not even discussed. Furthermore, you may not always have access to this text when you need to solve a problem. Below we demonstrate this second alternative.

EXAMPLE:

Malic acid is added to a solution which is already acidic. Initially

$$[H^+] = 1 \times 10^{-3} \text{ M}$$

$$[H_2Mal] = 5 \times 10^{-4} \text{ M}$$

K for malic acid is 4×10^{-4}.

Find the equilibrium $[H^+]$, $[HMal^-]$ and $[H_2Mal]$.

SOLUTION:

Step 1: Make a table and insert all given information.

		H^+	$HMal^-$	H_2Mal
Concentrations (M)	Initial	1×10^{-3}	0	5×10^{-4}
	Equilibrium	?	?	?

Comment: The initial $[HMal^-]$ is given, although it is not directly stated. Since there is no mention of any $HMal^-$ being present before the addition of H_2Mal, we assume it is zero initially.

Step 2: Let

$$z = \text{the increase in } [H^+]$$

Since there was already a $[H^+] = 1 \times 10^{-3}$ M before the addition of H_2Mal, the equilibrium $[H^+]$ will be (1×10^{-3}) + (the increase in $[H^+]$) or

$$\text{equilibrium } [H^+] = (1 \times 10^{-3}) + z \text{ M}$$

Step 3: Use the equilibrium equation to write an expression for the equilibrium [anion].

$$H_2Mal \rightleftharpoons H^+ + HMal^-$$

The reaction states that each additional H^+ produced by the dissociation of H_2Mal also produces an $HMal^-$ ion. Therefore

$$z = \text{equilibrium } [HMal^-]$$

Step 4: To derive an expression for the equilibrium $[H_2Mal]$, use the idea that at all times

$$[HMal^-] + [H_2Mal] = \text{initial } [H_2Mal]$$

Substitute z for the equilibrium $[HMal^-]$ and the given value for the initial $[H_2Mal]$ into the above equation.

$$\text{equilibrium } [HMal^-] + \text{equilibrium } [H_2Mal] = \text{initial } [H_2Mal]$$

$$z + \text{equilibrium } [H_2Mal] = 5 \times 10^{-4}$$

$$\text{equilibrium } [H_2Mal] = (5 \times 10^{-4}) - z$$

Step 5: Insert the expressions for the equilibrium concentrations into the table.

		H^+	$HMal^-$	H_2Mal
Concentrations (M)	Initial	1×10^{-3}	0	5×10^{-4}
	Equilibrium	$(1 \times 10^{-3}) + z$	z	$(5 \times 10^{-4}) - z$

Step 6: Substitute the expressions for the equilibrium concentrations into the dissociation constant equation and solve for z.

$$K = \frac{[H^+][HMal^-]}{[H_2Mal]}$$

$$4 \times 10^{-4} = \frac{(10^{-3} + z)(z)}{(5 \times 10^{-4}) - z}$$

The solution of the resulting quadratic equation is

$$z \approx 1.3 \times 10^{-4}$$

Step 7: Substitute the numerical value for z into the expressions for the equilibrium concentrations.

$$\text{equilibrium } [H^+] = (1 \times 10^{-3}) + z$$
$$\approx (1 \times 10^{-3}) + (1.3 \times 10^{-4})$$
$$\approx (1 \times 10^{-3}) + (.13 \times 10^{-3})$$
$$\approx 1.13 \times 10^{-3} \text{ M}$$

$$\text{equilibrium } [HMal^-] = z$$
$$\approx 1.3 \times 10^{-4} \text{ M}$$

$$\text{equilibrium } [H_2Mal] = (5 \times 10^{-4}) - z$$
$$\approx (5 \times 10^{-4}) - (1.3 \times 10^{-4})$$
$$\approx 3.7 \times 10^{-4} \text{ M}$$

PROBLEM SET 19:

For Problems 1 through 5 convert the number in standard decimal form to scientific notation.

1. 149,000,000 (distance from earth to sun in km)

2. 4,600,000,000 (age of earth in years)

3. 1,060,000,000,000 (volume of earth in km^3)

4. .00000000000000000000003 cm^3 (approximate amount of space occupied by one water molecule)

5. 30,000,000,000,000 (number of red blood cells in your body)

For Problems 6 through 10 convert the number in scientific notation into standard decimal form.

6. 6×10^{10} [number of white blood cells (leukocytes) in your body]

7. 3×10^{9} (approximate number of times your heart will beat in your lifetime)

8. 1.4×10^{16} (number of H^+ ions in a glass of water)

9. 1.5×10^{22} (number of cholesterol molecules in typical person)

10. 3.17×10^{-8} (fraction of one year represented by one second)

For Problems 11 through 13 give the corresponding pH.

11. $[H^+] = 1.25 \times 10^{-3}$ M

12. $[H^+] = 7.9 \times 10^{-6}$ M

13. $[H^+] = 5 \times 10^{-9}$ M

For Problems 14 through 16 give the corresponding $[H^+]$.

14. pH = 6.4

15. pH = 9.8

16. pH = .4

For Problems 17 through 19 determine whether a precipitate will form. Show calculations.

17. K_{sp} for $PbSO_4 = 1 \times 10^{-8}$

    $[Pb^{++}] = 10^{-3}$ M, $[SO_4^{--}] = 2 \times 10^{-4}$ M

18. K_{sp} for $BaSO_4 = 1.6 \times 10^{-10}$

    $[Ba^{++}] = 3 \times 10^{-6}$ M, $[SO_4^{--}] = 8 \times 10^{-5}$ M

19. K_{sp} for KHTar $= 4 \times 10^{-4}$

    $[K^+] = .1 \times 10^{-2}$, $[HTar^-] = 7 \times 10^{-2}$

20. Enough silver hydroxide (AgOH) is added to water to make a saturated solution. $K_{sp} = 1.5 \times 10^{-8}$. Find the equilibrium $[Ag^+]$ and $[OH^-]$.

21. Magnesium carbonate ($MgCO_3$) is added to water and forms a saturated solution. $K_{sp} = 2.5 \times 10^{-5}$. Find the equilibrium $[Mg^{++}]$ and $[CO_3^{--}]$.

22. H_2Tar is added to water. K, the dissociation constant for H_2Tar, is 1×10^{-3}. The initial $[H_2Tar] = 2 \times 10^{-3}$ M. Find the equilibrium $[H^+]$, $[HTar^-]$ and $[H_2Tar]$.

23. Succinic acid (H_2Suc) is added to an acidic solution (initial $[H^+] = 5.6 \times 10^{-5}$ M) in an amount sufficient to achieve an initial $[H_2Suc] = 1 \times 10^{-4}$ M. Determine the equilibrium $[H^+]$, $[HSuc^-]$ and $[H_2Suc]$. $K = 6.4 \times 10^{-5}$

24. What initial concentration of Malic acid (H_2Mal, $K = 4 \times 10^{-4}$) is required to achieve an equilibrium pH = 3?

25. Salicylic acid (a close relative of aspirin) has a dissociation constant of about 1×10^{-3}. H_2Sal (our abbreviation for salicylic acid) is added to an acidic solution (pH = 2.7) in an amount sufficient to create an initial $[H_2Sal] = 4 \times 10^{-3}$ M. Find the equilibrium $[H^+]$, $[HSal^-]$ and $[H_2Sal]$.

26. What initial concentration of salicylic acid (H_2Sal) is required to achieve an equilibrium pH = 2 in water? $K = 1 \times 10^{-3}$.

27. Pyruvic acid (HPyr) is added to water. K for HPyr is 3.2×10^{-3}. The initial $[HPyr] = 2.4 \times 10^{-3}$ M. Find the equilibrium $[H^+]$, $[Pyr^-]$ and [HPyr].

SECTION 20: COMPLEX NUMBERS

20-1 A Fable With a Moral

Once upon a time grade-school children were taught that it was impossible to take a larger number away from a smaller one. For example, they were taught that it was impossible to take away seven apples from a collection of only six apples. Today young children are no longer taught this limited view of subtraction. As you now know, there is a way to imagine a situation in which taking seven apples away from six can make sense. Suppose Elmo had six apples and owed Elma seven apples. Conceivably, he could borrow an apple from an apple vendor, give seven apples to Elma and end up being one apple in debt to the apple vendor. We can represent this transaction mathematically by the equation

$$6 - 7 = -1$$

where -1 is interpreted as being one apple in the red.

Historically, in the development of mathematics, the idea of negative numbers was not easily accepted by cultures that could not imagine a use for them. For the ancient Greeks, mathematics was practically equivalent to geometry. They could not visualize the possibility of subtracting a 7-cm line from a 6-cm line. The Hindus were more algebraically oriented and had no trouble accepting the existence of negative numbers. Likewise, the Chinese gave official recognition to negative numbers more than two thousand years ago.

The moral to this story is this. Whenever it appears that it is "impossible" to solve a problem mathematically, perhaps the failure is a lack of imagination instead of a shortcoming in the possibilities of mathematics. The time has come to consider an example of such a problem.

20-2 $\sqrt{-1}$

Up to this point we have ignored the situation in which a square root of a negative number is indicated. However this situation occurs quite naturally when dealing with quadratic equations.

Consider the equation

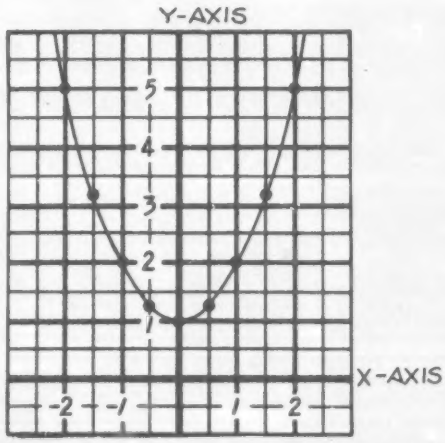
$$x^2 + 1 = 0$$

or, equivalently,

$$x^2 = -1$$

In other words, we are looking for a number, x, such that $x^2 = -1$. We can very quickly elminate two candidates for x; x is not 1, because 1^2 is 1, not -1. Similarly, x is not -1, because $(-1)^2$ is 1, not -1.

Recall that we can graph a quadratic equation and find the roots of the equation from the graph. We can try this approach to decipher the meaning of $x^2 = -1$. To the right we have a graph of $y = x^2 + 1$.



Now all we have to do is look at the curve to find out what x is when y = 0. This occurs when the curve intercepts the x-axis. But the curve does not touch the x-axis. Again we are stymied. It appears to be impossible to solve for x in the equation $x^2 = -1$.

However, we hark back to the moral stated earlier. Perhaps we only lack imagination. One approach we can try is to create a new number i and define it so that

$$i^2 = -1$$

and therefore

$$i = \sqrt{-1}$$

You may feel dubious about this new mathematical creature. The people who first played with the idea of i certainly felt dubious about it. They felt sufficiently skeptical about it to call it an "imaginary number."

Since those early times enough uses for i have been found that the title "imaginary" is a historical relic. The usefulness of i is related to its ability to solve mathematical problems. Later on in this course we will use i in this fashion. For now, though, we will content ourselves with a brief exploration of the world of numbers that includes imaginary numbers. This set of numbers is called the complex numbers. A complex number is a number which can be written in the form a + bi with "a" and "b" real numbers. For example, 3 + 5i, 0 + 2i or just 2i, and 8 + 0i or just 8, are all complex numbers. When the value of b is zero, as in the case 8 + 0i, then the complex number is also a real number.

20-3 Complex Numbers

Let's look at another quadratic equation that has complex roots.

$$0 = x^2 - 2x + 5$$

The quadratic formula says

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case, a = 1, b = -2, and c = 5. Therefore,

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \cdot 1 \cdot 5)}}{2}$$

$$x = 1 \pm \frac{\sqrt{-16}}{2}$$

$$x = 1 \pm \frac{\sqrt{(16)(-1)}}{2}$$

Since 16 is a perfect square we can remove it from under the radical sign.

$$x = 1 \pm \frac{4\sqrt{-1}}{2}$$

$$= 1 \pm 2\sqrt{-1}$$

In this section we have defined

$$\sqrt{-1} = i;$$

therefore,

$$x = 1 \pm 2i$$

This means, as you know by now, that both

$$x = 1 + 2i$$

and

$$x = 1 - 2i$$

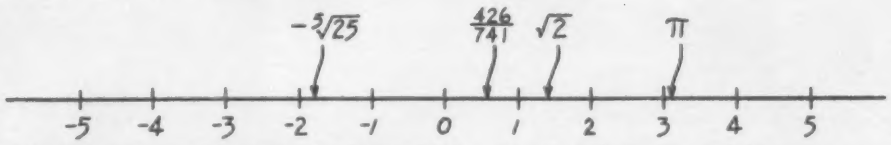
are roots of the equation

$$0 = x^2 - 2x + 5$$

The numbers 1 + 2i and 1 - 2i are both examples of complex numbers.

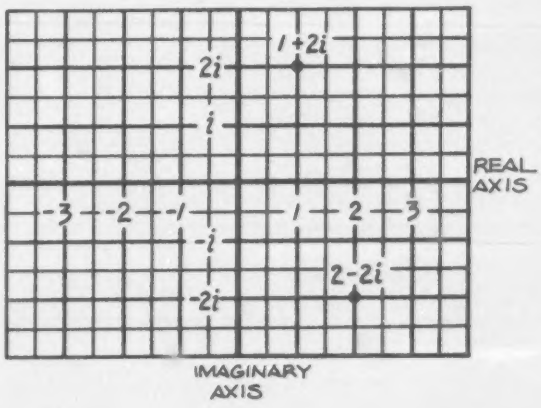
20-4 Graphs of Complex Numbers

Complex numbers which have no imaginary part are called real numbers. Real numbers are just all of the numbers that you have dealt with so far that have not included the square roots of negative numbers. All of the familiar numbers like $\sqrt{2}$, $\frac{426}{741}$, $5\sqrt{25}$, π and so forth, have been real numbers. All real numbers may be thought of as addresses of points on a line.



This means that we can graph any real number as a point on a line.

To graph complex numbers we need more than a line; we need a plane. The graph on the right demonstrates how complex numbers may be assigned to points on a plane. The complex number 1 + 2i is graphed as the point (1, 2). The real part of the complex number becomes the horizontal coordinate. The coefficient of i (in this case 2) becomes the vertical coordinate. In this scheme every complex number can be plotted as a point on the plane and every point on the plane corresponds to some complex number. Graphing complex numbers is an aid in solving many mathematical problems.



PROBLEM SET 20:

Answer true or false to Problems 1 through 10 by writing T for true and F for false.

1. It is impossible to make sense out of taking a million apples away from a collection of zero apples.

2. Imagination has no place in mathematics.

3. It is impossible to solve the equation $x^2 = -4$.

4. When we include imaginary numbers in our number system, all quadratic equations have solutions.

5. $1 + i$ is a complex number.

6. An imaginary number is a number that is completely without use.

7. $\frac{\pi}{\sqrt{3}}$ is a complex number.

8. We can graph all complex numbers as points on a number line.

9. The equation $x^2 + 1 = 0$ is a quadratic equation with complex roots.

10. Both of the following are true

$$i = \sqrt{-1}$$

$$i^2 = -1$$

Graph the complex numbers listed in Problems 11 through 15.

11. $-4 - 2i$

12. $3 - i$

13. $-1.5 + i$

14. $4 + 3i$

15. $\pi + \pi i$

In Problems 16 through 21 find the roots of the quadratic equation.

16. $x^2 + 2x + 2 = 0$

17. $x^2 - 6x + 13 = 0$

18. $x^2 + 2x + 5 = 0$

19. $x^2 - 4x + 5 = 0$

20. $x^2 + 4x + 8 = 0$

21. $x^2 + 6x + 10 = 0$

SECTION 21: OPERATIONS WITH COMPLEX NUMBERS

21-1 Addition, Subtraction and Multiplication

Now that we have defined complex numbers, it is reasonable to wonder whether their behavior is similar to the behavior of the more familiar numbers. In other words, can they be added, subtracted, multiplied and divided just as real numbers can? This may seem like a trivial question now. Be warned that it is not. As we start out our exploration of the arithmetic of complex numbers it will seem very simple, but there will be surprising twists in the story later on.

ADDITION PROBLEM:

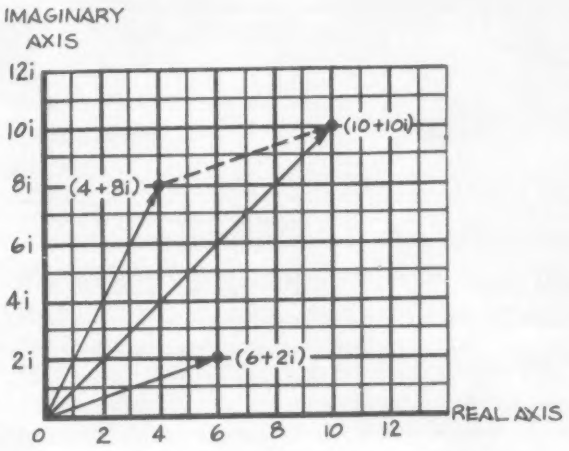
Find the sum $(2 + 3i) + (9 + 7i)$.

SOLUTION:

We pretend that the two complex numbers are simple binomials and proceed.

$$\begin{array}{r} 6 + 2i \\ 4 + 8i \\ \hline 10 + 10i \end{array}$$

Graphically, the process of complex addition is identical to vector addition.



SUBTRACTION PROBLEM:

Find the difference:

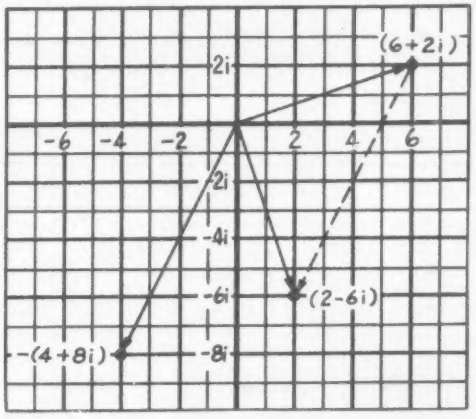
$$\begin{array}{r} (6 + 2i) \\ - \ (4 + 8i) \\ \hline ? \quad ? \end{array}$$

SOLUTION:

Again we treat this problem just like a problem in binomial subtraction.

$$(6 + 2i) - (4 + 8i) = 6 + 2i - 4 - 8i$$
$$= 2 - 6i$$

Again, when the process of complex subtraction is represented graphically it is similar to subtraction of vectors.



MULTIPLICATION PROBLEM:

$(1 + i)(2 - 3i) = ?$

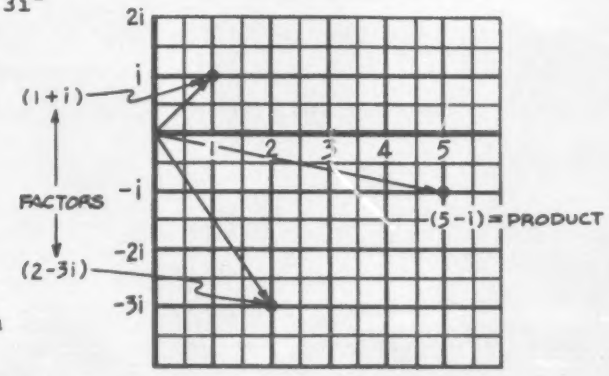
SOLUTION:

Again, we pretend that the complex numbers are simple binomials.

$$\begin{array}{r} 2 - 3i \\ \underline{1 + \ \ i} \\ 2i - 3i^2 \\ \underline{2 - 3i} \\ 2 - \ \ i - 3i^2 \end{array}$$

Next we use the relation $i^2 = -1$.

$$\begin{aligned} (2 - 3i)(1 + i) &= 2 - i - 3i^2 \\ &= 2 - i - 3(-1) \\ &= 2 - i + 3 \\ &= 5 - i \end{aligned}$$



The process of complex multiplication is <u>not similar</u> to any vector operation, as the graph to the right illustrates.

The operation of division presents unexpected difficulties. We will deal with this operation in the next section.

21-2 <u>Powers of i</u>

We can use the relation $i^2 = -1$ to see how to simplify higher powers of i. For example, consider i^3.

$$\begin{aligned} i^3 &= (i^2)(i) \\ &= (-1)(i) \\ &= -i \end{aligned}$$

Similarly,

$$\begin{aligned} i^4 &= (i^2)(i^2) \\ &= (-1)(-1) \\ &= 1 \end{aligned}$$

The relations listed below may be used to simplify higher powers of i.

$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

<u>EXAMPLE</u>:

Simplify i^{23}.

<u>SOLUTION</u>:

The first step is to use the identity $i^4 = 1$. Notice that

$$\begin{aligned} i^8 &= (i^4)(i^4) \\ &= 1 \end{aligned}$$
$$i^{12} = 1$$
$$i^{16} = 1$$

In general, any power of i that is divisible by 4 will be identical to 1. With this in mind, we write

$$i^{23} = (i^{20})(i^3)$$

Notice that 20 is divisible by 4. Therefore, $1^{20} = 1$, and

$$\begin{aligned} i^{23} &= 1^{20} \cdot i^3 \\ &= 1 \cdot (i^3) \\ &= 1 \cdot (-i) \\ &= -i \end{aligned}$$

PROBLEM SET 21:

Perform the indicated operations.

1. $(6 + 3i) + (2 - 12i) =$
2. $(-3 + i) + (8 + 3i) =$
3. $(-4 - 2i) + (5 - 6i) =$
4. $(\frac{1}{2} - 3i) + (\frac{3}{2} + 2i) =$
5. $(4 + 16i) - (3 + 7i) =$
6. $(1 - 2i) - (3 - 4i) =$
7. $(-2 + 5i) - (6 - i) =$
8. $(7 - i) - (-4 + 3i) =$
9. $(-4 - 6i) - (-5 - 8i) =$
10. $(3 + 2i)(2 + 6i) =$
11. $(-2 + 3i)(1 + i) =$
12. $(8 - 3i)(-3 + 9i) =$
13. $(10 + 5i)(2 - 3i) =$
14. $(-2 - 4i)(-3 - 5i) =$
15. $(2 + 3i)(2 - 3i) =$
16. $(-1 + 4i)(-1 - 4i) =$
17. $(1 + i)(1 + 2i)(1 + 3i) =$

For Problems 18 through 20 represent the arithmetic operations graphically.

18. $(3 - 4i) + (-6 + 2i)$
19. $(7 + 4i) - (1 - 6i)$
20. $(-5 + 4i)(3 + 2i)$

In Problems 21 through 27 reduce the exponent of i to 1 or eliminate i completely. Show your work.

21. $i^3 =$
22. $i^4 =$
23. $i^5 =$
24. $i^7 =$
25. $i^{24} =$
26. $i^{17} =$
27. $i^{14} =$

SECTION 22: DIVISION OF COMPLEX NUMBERS

22-1 Reciprocals of Complex Numbers

In the previous section we proceeded under the simple assumptions that complex numbers were binomials and that $i^2 = -1$. These assumptions were all that were needed to show how to add, subtract and multiply complex numbers. For those of you who remember how to divide polynomials, we have an example below that shows the dead-end of this approach. It shows that the rules that apply to polynomial division are of no help in complex division. This is optional. It is here for you to examine if you are interested.

PROBLEM:

$$\frac{1}{1+i} = ?$$

NON-SOLUTION:

$$\begin{array}{r|l} & 1 - i - 1 + i \\ \hline 1 + i & 1 \\ & \underline{1 + i} \\ & \quad -i \\ & \quad \underline{-i - i^2} \\ & \qquad + i^2 \\ & (=) \; -1 \\ & \qquad \underline{-1 - i} \\ & \qquad\quad + i \\ & \qquad\quad \underline{i + i^2} \\ & \qquad\qquad - i^2 \\ & \qquad\quad (=) \; 1 \end{array}$$

This pattern may be repeated forever. We cannot get the answer this way.

Now we'll show you a method that works. It depends on observing something about multiplication of complex numbers. What do all the following statements have in common?

$$(3 + 2i)(3 - 2i) = 13$$

$$(5 - i)(5 + i) = 26$$

$$(-2 + 4i)(-2 - 4i) = 20$$

There are two things to observe. First, the result of each multiplication is a real number; there is no imaginary part. Second, the two complex numbers in each multiplication have real parts that are equal and imaginary parts that are equal with opposite signs. For example 3 + 2i and 3 - 2i both have the real part 3. Note also that the imaginary parts, 2i and -2i, differ in sign. A pair of complex numbers related in this way are called complex conjugates. We say that 3 + 2i is the conjugate of 3 - 2i, and similarly that 3 - 2i is the conjugate of 3 + 2i.

When a pair of complex conjugates are multiplied, the result is always a real number.

EXAMPLE:

Find the result when 3 - 5i is multiplied by its conjugate.

SOLUTION:

The conjugate of $3 - 5i$ is $3 + 5i$. Multiplying, we get

$$(3 - 5i)(3 + 5i) = 9 - 15i + 15i - 25i^2$$

$$= 34$$

Now we return to the problem posed at the beginning of this section. We compute the reciprocal of $1 + i$. The method which works is to multiply the numerator and denominator of the fraction by $1 - i$, the conjugate of $1 + i$.

$$\frac{1}{1 + i} = \frac{1}{1 + i} \cdot \frac{1 - i}{1 - i}$$

Notice that the value of the fraction is not changed since $\frac{1 - i}{1 - i}$ is just 1. We continue.

$$\frac{1}{1 + i} = \frac{1 - i}{(1 + i)(1 - i)}$$

$$= \frac{1 - i}{2}$$

$$= \frac{1}{2} - \frac{1}{2}i$$

22-2 Divison of Complex Numbers

The process just illustrated is the key to the process of division of complex numbers.

PROBLEM:

$$\frac{4 + 2i}{1 + i} = ?$$

SOLUTION:

We know that the division problem

$$\frac{4 + 2i}{1 + i} = 4 + 2i \cdot \frac{1}{1 + i}$$

From the previous section we know that

$$\frac{1}{1 + i} = \frac{1}{2} - \frac{1}{2}i$$

therefore,

$$\frac{4 + 2i}{1 + i} = (4 + 2i)(\frac{1}{2} - \frac{1}{2}i).$$

By carrying out the indicated operation on the right we determine the quotient of the expression on the left,

$$\frac{4 + 2i}{1 + i} = 2 + i - 2i - i^2$$

$$= 2 - i - (-1)$$

$$= 2 - i + 1$$

$$= 3 - i$$

and we are done.

PROBLEM SET 22:

Find the conjugate of each complex number.

1. $3 - 2i$
2. $8 + 6i$
3. $100 - 30i$
4. $-10 + 7i$
5. $-15 - 19i$
6. $6i$

Find the following reciprocals.

7. $\frac{1}{2 - i}$
8. $\frac{1}{1 + 3i}$
9. $\frac{1}{1 - i}$
10. $\frac{1}{1 + 2i}$
11. $\frac{1}{2 + 2i}$
12. $\frac{1}{1 - 2i}$
13. $\frac{1}{3 + 3i}$
14. $\frac{1}{6i}$

Find the following quotients.

15. $\frac{7 - i}{3 + i}$
16. $\frac{-10 - 20i}{-7 + i}$
17. $\frac{6 - 4i}{5 + i}$
18. $\frac{-7 + 6i}{1 + 2i}$
19. $\frac{8 - 4i}{2 + 2i}$
20. $\frac{11 + 3i}{1 - 2i}$
21. $\frac{12i}{3 + 3i}$
22. $\frac{-60 - 6i}{6i}$

n	$\sqrt{n}$	n	$\sqrt{n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
		50	7.071	100	10.00	31.62	150	12.25	38.73	200	14.14	44.72	250	15.81	50.00
1	1.000	51	7.141	101	10.05	31.78	151	12.29	38.86	201	14.18	44.83	251	15.84	50.10
2	1.414	52	7.211	102	10.10	31.94	152	12.33	38.99	202	14.21	44.94	252	15.87	50.20
3	1.732	53	7.280	103	10.15	32.09	153	12.37	39.12	203	14.15	45.06	253	15.91	50.30
4	2.000	54	7.348	104	10.20	32.25	154	12.41	39.24	204	14.28	45.17	254	15.94	50.40
5	2.236	55	7.416	105	10.25	32.40	155	12.45	39.37	205	14.32	45.28	255	15.97	50.50
6	2.449	56	7.483	106	10.30	32.56	156	12.49	39.50	206	14.35	45.39	256	16.00	50.60
7	2.646	57	7.550	107	10.34	32.71	157	12.53	39.62	207	14.39	45.50	257	16.03	50.70
8	2.828	58	7.616	108	10.39	32.86	158	12.57	39.75	208	14.42	45.61	258	16.06	50.79
9	3.000	59	7.681	109	10.44	33.01	159	12.61	39.87	209	14.46	45.72	259	16.09	50.89
10	3.162	60	7.746	110	10.49	33.17	160	12.65	40.00	210	14.49	45.83	260	16.12	50.99
11	3.317	61	7.810	111	10.54	33.32	161	12.69	40.12	211	14.52	45.93	261	16.16	51.09
12	3.464	62	7.874	112	10.58	33.47	162	12.73	40.25	212	14.56	46.04	262	16.19	51.19
13	3.606	63	7.937	113	10.63	33.62	163	12.77	40.37	213	14.59	46.15	263	16.22	51.28
14	3.742	64	8.000	114	10.68	33.76	164	12.81	40.50	214	14.63	46.26	264	16.25	51.38
15	3.873	65	8.062	115	10.72	33.91	165	12.85	40.62	215	14.66	56.37	265	16.28	51.48
16	4.000	66	8.124	116	10.77	34.06	166	12.88	40.74	216	14.70	46.48	266	16.31	51.58
17	4.123	67	8.185	117	10.82	34.21	167	12.92	40.87	217	14.73	46.58	267	16.34	51.67
18	4.243	68	8.246	118	10.86	34.35	168	12.96	40.99	218	14.76	46.69	268	16.37	51.77
19	4.359	69	8.307	119	10.91	34.50	169	13.00	41.11	219	14.80	46.80	269	16.49	51.87
20	4.472	70	8.367	120	10.95	34.64	170	13.04	41.23	220	14.83	46.90	270	16.43	51.96
21	4.583	71	8.426	121	11.00	34.79	171	13.08	41.35	221	14.87	47.01	271	16.46	52.06
22	4.690	72	8.485	122	11.04	34.93	172	13.11	41.47	222	14.90	47.12	272	16.49	52.14
23	4.796	73	8.544	123	11.09	35.07	173	13.15	41.59	223	14.93	47.22	273	16.52	52.25
24	4.899	74	8.602	124	11.13	35.21	174	13.19	41.71	224	14.97	47.33	274	16.55	52.35
25	5.000	75	8.660	125	11.18	35.36	175	13.23	41.83	225	15.00	47.43	275	16.58	52.44
26	5.099	76	8.718	126	11.22	35.50	176	13.27	41.95	226	15.03	47.54	276	16.61	52.54
27	5.196	77	8.775	127	11.27	35.64	177	13.30	42.07	227	15.07	47.64	277	16.64	52.63
28	5.292	78	8.832	128	11.31	35.78	178	13.34	42.19	228	15.10	47.75	278	16.67	52.73
29	5.385	79	8.888	129	11.36	35.92	179	13.38	42.31	229	15.13	47.85	279	16.70	52.82
30	5.477	80	8.944	130	11.40	36.06	180	13.42	42.43	230	15.17	47.96	280	16.73	52.92
31	5.568	81	9.000	131	11.45	36.19	181	13.45	42.54	231	15.29	48.96	281	16.76	53.01
32	5.657	82	9.055	132	11.49	36.33	182	13.49	42.66	232	15.23	48.17	282	16.79	53.10
33	5.745	83	9.110	133	11.53	36.47	183	13.53	42.78	233	15.26	58.27	283	16.82	53.20
34	5.831	84	9.165	134	11.58	36.61	184	13.56	42.90	234	15.30	58.37	284	16.84	53.29
35	5.916	85	9.220	135	11.62	36.74	185	13.60	43.01	235	15.33	48.48	285	16.88	53.39
36	6.000	86	9.274	136	11.66	36.88	186	13.64	43.13	236	15.36	48.58	286	16.91	53.48
37	6.083	87	9.327	137	11.70	37.01	187	13.67	43.24	237	15.39	58.68	287	16.94	53.57
38	6.164	88	9.381	138	11.75	37.15	188	13.71	43.36	238	15.43	48.79	288	16.97	53.67
39	6.245	89	9.434	139	11.79	37.28	189	13.75	43.47	239	15.46	48.89	289	17.00	53.76
40	6.325	90	9.487	140	11.83	37.42	190	13.78	43.59	240	15.49	48.99	290	17.03	53.85
41	6.403	91	9.539	141	11.87	37.55	191	13.82	43.70	241	15.52	49.09	291	17.06	53.94
42	6.481	92	9.592	142	11.92	37.68	192	13.86	43.82	242	15.56	59.19	292	17.09	54.04
43	6.557	93	9.644	143	11.96	37.82	193	13.89	43.93	243	15.59	49.30	293	17.12	54.13
44	6.633	94	9.695	144	12.00	37.95	194	13.93	44.05	244	15.62	49.40	294	17.15	54.22
45	6.708	95	9.747	145	12.04	38.08	195	13.96	44.16	245	15.65	49.50	295	17.18	54.31
46	6.782	96	9.798	146	12.08	38.21	196	14.00	44.27	246	15.68	49.60	296	17.20	54.41
47	6.856	97	9.849	147	12.12	38.34	197	14.04	44.38	247	15.72	49.70	297	17.23	54.50
48	6.928	98	9.899	148	12.16	38.47	198	14.07	44.50	248	15.75	49.80	298	17.26	54.59
49	7.000	99	9.950	149	12.21	38.60	199	14.11	44.61	249	15.78	59.90	299	17.29	54.68

n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
300	17.32	54.77	350	18.71	59.16	400	20.00	63.25	450	21.21	67.08	500	22.36	70.71
301	17.35	54.86	351	18.73	59.25	401	20.02	63.32	451	21.24	67.16	501	22.38	70.78
302	17.38	54.95	352	18.76	59.33	402	20.05	63.40	452	21.26	67.23	502	22.41	70.85
303	17.40	55.05	353	18.79	59.41	403	20.07	63.48	453	21.28	67.31	503	22.43	79.92
304	17.44	55.14	354	18.81	59.50	404	20.10	63.56	454	21.31	67.38	504	22.45	70.99
305	17.46	55.23	355	18.84	59.58	405	20.12	63.64	455	21.33	67.45	505	22.47	71.06
306	17.49	55.32	356	18.87	59.67	406	20.15	63.72	456	21.35	67.53	506	22.49	71.13
307	17.52	55.41	357	18.89	59.75	407	20.17	63.80	567	21.38	67.60	507	22.52	71.20
308	17.55	55.50	358	18.92	59.83	408	20.20	63.87	458	21.40	67.68	508	22.54	71.27
309	17.58	55.59	359	18.95	59.92	409	20.22	63.95	459	21.42	67.75	509	22.56	71.34
310	17.61	55.68	360	18.97	60.00	410	20.25	64.03	460	21.45	67.82	510	22.58	71.41
311	17.64	55.77	361	19.00	60.08	411	20.27	64.11	461	21.47	67.90	511	22.61	71.48
312	17.66	55.86	362	19.03	60.17	412	20.30	64.19	462	21.49	67.97	512	22.63	71.55
313	17.69	55.95	363	19.05	60.25	413	20.32	64.27	463	21.52	68.04	513	22.65	71.62
314	17.72	56.04	364	19.08	60.33	414	20.35	64.34	464	21.54	68.12	514	22.67	71.69
315	17.75	56.12	365	19.10	60.42	415	20.37	64.42	465	21.56	68.19	515	22.69	71.76
316	17.78	56.21	366	19.13	60.50	416	20.40	64.50	466	21.59	68.26	516	22.72	71.83
317	17.80	56.30	367	19.16	60.58	417	20.42	64.58	467	21.61	68.34	517	22.74	71.90
318	17.83	56.39	368	19.18	60.66	418	20.45	64.65	468	21.63	68.41	418	22.76	71.97
319	17.86	56.48	369	19.21	60.75	419	20.47	64.73	469	21.66	68.48	519	22.78	72.04
320	17.89	56.57	370	19.24	60.83	420	20.49	64.81	470	21.68	68.56	520	22.80	72.11
321	17.92	56.66	371	19.26	60.91	421	20.52	64.88	471	21.70	68.63	521	22.83	72.18
322	17.94	56.75	372	19.29	60.99	422	20.54	64.96	472	21.73	68.70	522	22.85	72.25
323	17.97	56.83	373	19.31	61.07	423	20.57	65.04	473	21.75	68.78	523	22.87	72.32
324	18.00	56.92	374	19.34	61.16	424	20.59	65.12	474	21.77	68.85	524	22.89	72.39
325	18.03	57.01	375	19.36	61.24	425	20.62	65.19	475	21.79	68.92	525	22.91	72.46
326	18.06	57.10	376	19.39	61.32	426	20.64	65.27	476	21.82	68.99	526	22.93	72.53
327	18.08	57.18	377	19.42	61.40	427	20.66	65.35	477	21.84	69.07	527	22.96	72.59
328	18.11	57.27	378	19.44	61.48	428	20.69	65.42	478	21.86	69.14	528	22.98	72.66
329	18.14	57.36	379	19.47	61.56	429	20.71	65.50	479	21.89	69.21	529	23.00	72.73
330	18.17	57.45	380	19.49	61.64	430	20.74	65.57	480	21.91	69.28	530	23.02	72.80
331	18.19	57.53	381	19.52	61.73	431	20.76	65.65	481	21.93	69.35	531	23.04	72.87
332	18.22	57.62	382	19.54	61.81	432	20.78	65.73	482	21.95	69.43	532	23.07	72.94
333	18.25	57.71	383	19.57	61.89	433	20.81	65.80	483	21.98	69.50	533	23.09	73.01
334	18.28	57.79	384	19.60	61.97	434	20.83	65.88	484	22.00	69.57	534	23.11	73.08
335	18.30	57.88	385	19.62	62.05	435	20.86	65.95	485	22.02	69.64	535	23.13	73.14
336	18.33	57.97	386	19.65	62.13	436	20.88	66.03	486	22.05	69.71	536	23.15	73.21
337	18.36	58.05	387	19.67	62.21	437	20.90	66.11	487	22.07	69.79	537	23.17	73.28
338	18.38	58.14	388	19.70	62.29	438	20.93	66.18	488	22.09	69.86	538	23.19	73.35
339	18.41	58.22	389	19.72	62.37	439	20.95	66.26	489	22.11	69.93	539	23.22	73.42
340	18.44	58.31	390	19.75	62.45	440	20.98	66.33	490	22.14	70.00	540	23.24	73.48
341	18.47	58.40	391	19.77	62.53	441	21.00	66.41	491	22.16	70.07	541	23.26	73.55
342	18.49	58.48	392	19.80	62.61	442	21.02	66.48	492	22.18	70.14	542	23.28	73.62
343	18.52	58.57	393	19.82	62.69	443	21.05	66.56	493	22.20	70.21	543	23.30	73.69
344	18.55	58.65	394	19.85	62.77	444	21.07	66.63	494	22.23	70.29	544	23.32	73.76
345	18.57	58.74	395	19.87	62.85	445	21.10	66.71	495	22.25	70.36	545	23.35	73.82
346	18.60	58.82	396	19.90	62.93	446	21.12	66.78	596	22.27	70.43	546	23.37	73.89
347	18.63	58.91	397	19.92	63.01	447	21.14	66.86	497	22.29	70.50	547	23.39	73.96
348	18.65	58.99	398	19.95	63.09	448	21.17	66.93	498	22.32	70.57	548	23.41	74.03
349	18.68	59.08	399	19.97	63.17	449	21.19	67.01	499	22.34	70.64	549	23.43	74.09

n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
550	23.45	74.16	600	24.49	77.46	650	25.50	80.62	700	26.46	83.67	750	27.39	86.60
551	23.47	74.23	601	24.52	77.52	651	25.51	80.68	701	26.48	83.73	751	27.40	86.66
552	23.49	74.30	602	24.54	77.59	652	25.53	80.75	702	26.50	83.79	752	27.42	86.72
553	23.52	74.36	603	24.56	77.65	653	25.55	80.81	703	26.51	83.85	753	27.44	86.78
554	23.54	74.43	604	24.58	77.72	654	25.57	80.87	704	26.53	83.90	754	27.46	86.83
555	23.56	74.50	605	24.60	77.78	655	25.59	80.93	705	26.55	83.96	755	27.48	86.89
556	23.58	74.57	606	24.62	77.85	656	25.61	80.99	706	26.57	84.02	756	27.50	86.95
557	23.60	74.63	607	24.64	77.91	657	25.63	81.06	707	26.59	84.08	757	27.51	87.01
558	23.62	74.70	608	24.66	77.97	658	25.65	81.12	708	26.61	84.14	758	27.53	87.06
559	23.64	74.77	609	24.68	78.04	659	25.67	81.18	709	26.63	84.20	759	27.55	87.12
560	23.66	74.83	610	24.70	78.10	660	25.69	81.24	710	26.65	84.26	760	27.57	87.18
561	23.69	74.90	611	24.72	78.17	661	25.71	81.30	711	26.66	84.32	761	27.59	87.24
562	23.71	74.97	612	24.74	78.23	662	25.73	81.36	712	26.68	84.38	762	27.60	87.29
563	23.73	75.03	613	24.76	78.29	663	25.75	81.42	713	26.70	84.44	763	27.62	87.35
564	23.75	75.10	614	24.78	78.36	664	25.77	81.49	714	26.72	84.50	764	27.64	87.41
565	23.77	75.17	615	24.80	78.42	665	25.79	81.55	715	26.74	84.56	765	27.66	87.46
566	23.79	75.23	616	24.82	78.49	666	25.81	81.61	716	26.76	84.62	766	27.68	87.52
567	23.81	75.30	617	24.84	78.55	667	25.83	81.67	717	26.78	84.68	767	27.69	87.58
568	23.83	75.37	618	24.86	78.61	668	25.85	81.73	718	26.80	84.73	768	27.71	87.64
569	23.85	75.43	619	24.88	78.68	669	25.87	81.79	719	26.81	84.79	769	27.73	87.69
570	23.87	75.50	620	24.90	78.74	670	25.88	81.85	720	26.83	84.85	770	27.75	87.75
571	23.90	75.56	621	24.92	78.80	671	25.90	81.91	721	26.85	84.91	771	27.77	87.81
572	23.92	75.63	622	24.94	78.87	672	25.92	81.98	722	26.87	84.97	772	27.78	87.86
573	23.94	75.70	623	24.96	78.93	673	25.94	82.04	723	26.89	85.03	773	27.80	87.92
574	23.96	75.76	624	24.98	78.99	674	25.96	82.10	724	26.91	85.09	774	27.82	87.98
575	23.98	75.83	625	25.00	79.06	675	25.98	82.16	725	26.93	85.15	775	27.84	88.03
576	24.00	75.89	626	25.02	79.12	676	26.00	82.22	726	26.94	85.21	776	27.86	88.09
577	24.02	75.96	627	25.04	79.18	677	26.02	82.28	727	26.96	85.26	777	27.87	88.15
578	24.04	76.03	628	25.06	79.25	678	26.04	82.34	728	26.98	85.32	778	27.89	88.20
579	24.06	76.09	629	25.08	79.31	679	26.06	82.40	729	27.00	85.38	779	27.91	88.26
580	24.08	76.16	630	25.10	79.37	680	26.08	82.46	730	27.02	85.44	780	27.93	88.32
581	24.10	76.22	631	25.12	79.44	681	26.10	82.52	731	27.04	85.50	781	27.95	88.37
582	24.12	76.29	632	25.14	79.50	682	26.12	82.58	732	27.06	85.56	782	27.96	88.43
583	24.15	76.35	633	25.16	79.56	683	26.13	82.64	733	27.07	85.62	783	27.98	88.49
584	24.17	76.42	634	25.18	79.62	684	26.15	82.70	734	27.09	85.67	784	28.00	88.54
585	24.19	76.49	635	25.20	79.69	685	26.17	82.76	735	27.11	85.73	785	28.02	88.60
586	24.21	76.55	636	25.22	79.75	686	26.19	82.83	736	27.13	85.79	786	28.04	88.66
587	24.23	76.62	637	25.24	79.81	687	26.21	82.89	737	27.15	85.85	787	28.05	88.71
588	24.25	76.68	638	25.26	79.87	688	26.23	82.95	738	27.17	85.91	788	28.07	88.77
589	24.27	76.75	639	25.28	79.94	689	26.25	83.01	739	27.18	85.97	789	28.09	88.83
590	24.29	76.81	640	25.30	80.00	690	26.27	83.07	740	27.20	86.02	790	28.11	88.88
591	24.31	76.88	641	25.32	80.06	691	26.29	83.13	741	27.22	86.08	791	28.12	88.94
592	24.33	76.94	642	25.34	80.12	692	26.31	83.19	742	27.24	86.14	792	28.14	88.99
593	24.35	77.01	643	25.36	80.19	693	26.32	83.25	743	27.26	86.20	793	28.16	89.05
594	24.37	77.07	644	25.38	80.25	694	26.34	83.31	744	27.28	86.26	794	28.18	89.11
595	24.39	77.14	645	25.40	80.31	695	26.36	83.37	745	27.29	86.31	795	28.20	89.16
596	24.41	77.20	646	25.42	80.37	696	26.38	83.43	746	27.31	86.37	796	28.21	89.22
597	24.43	77.27	647	25.44	80.44	697	26.40	83.49	747	27.33	86.43	797	28.23	89.27
498	24.45	77.33	648	25.46	80.50	698	26.42	83.55	748	27.35	86.49	798	28.25	89.33
599	24.47	77.40	649	25.48	80.56	699	26.44	83.61	749	27.37	86.54	799	28.27	89.39

n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
800	28.28	89.44	850	29.15	92.20	900	30.00	94.87	950	30.82	97.47
801	28.30	89.50	851	29.17	92.25	901	30.02	94.92	951	30.84	97.52
802	28.32	89.55	852	29.19	92.30	902	30.03	94.97	952	30.85	97.57
803	28.34	89.61	853	29.21	92.36	903	30.05	95.03	953	30.87	97.62
804	28.35	89.67	854	29.22	92.41	904	30.07	95.08	954	30.89	97.67
805	28.37	89.72	855	29.24	92.47	905	30.08	95.13	955	30.90	97.72
806	28.39	89.78	856	29.26	92.52	906	30.10	95.18	956	30.92	97.78
807	28.41	89.83	857	29.27	92.57	907	30.12	95.24	957	30.94	97.83
808	28.43	89.89	858	29.29	92.63	908	30.13	95.29	958	30.95	97.88
809	28.44	89.94	859	29.31	92.68	909	30.15	95.34	959	30.97	97.93
810	28.46	90.00	860	29.33	92.74	910	30.17	95.39	960	30.98	97.98
811	28.48	90.05	861	29.34	92.79	911	30.18	95.45	961	31.00	98.03
812	28.50	91.11	862	29.36	92.84	912	30.20	95.50	962	31.02	98.08
813	28.51	90.17	863	29.38	92.90	913	30.22	95.55	963	31.03	98.13
814	28.53	90.22	864	29.39	92.95	914	30.23	95.60	964	31.05	98.18
815	28.55	90.28	865	29.41	93.01	915	30.25	95.66	965	31.06	98.23
816	28.57	90.33	866	29.43	93.06	916	30.27	95.71	966	31.08	98.28
817	28.58	90.39	867	29.44	93.11	917	30.28	95.76	967	31.10	98.34
818	28.60	90.44	868	29.46	93.17	918	30.30	95.81	968	31.11	98.39
819	28.62	90.50	869	29.48	93.22	919	30.32	95.86	969	31.13	98.44
820	28.64	90.55	870	29.50	93.27	920	30.33	95.92	970	31.14	98.49
821	28.65	90.61	871	29.51	93.33	921	30.35	95.97	971	31.16	98.54
822	28.67	90.66	872	29.53	93.49	922	30.36	96.02	972	31.18	98.59
823	28.69	90.72	873	29.55	93.43	923	30.38	96.07	973	31.19	98.64
824	28.71	90.77	874	29.56	93.49	924	30.40	96.12	974	31.21	98.69
825	28.72	90.83	875	29.58	93.54	925	30.41	96.18	975	31.22	98.74
826	28.74	90.88	876	29.60	93.59	926	30.43	96.23	976	31.24	98.79
827	28.76	90.94	877	29.61	93.65	927	30.45	96.28	977	31.26	98.84
828	28.77	90.99	878	29.63	93.70	928	30.46	96.33	978	31.27	98.89
829	28.79	91.05	879	29.65	93.76	929	30.48	96.38	979	31.29	98.94
830	28.81	91.10	880	29.66	93.81	930	30.50	96.44	980	31.30	98.99
831	28.83	91.16	881	29.68	93.86	931	30.51	96.49	981	31.32	99.05
832	28.84	91.21	882	29.70	93.91	932	30.53	96.54	982	31.34	99.10
833	28.86	91.27	883	29.72	93.97	933	30.55	96.59	983	31.35	99.15
834	28.88	91.32	884	29.73	94.02	934	30.56	96.64	984	31.37	99.20
835	28.90	91.38	885	29.75	94.07	935	30.58	96.70	985	31.38	99.25
836	28.91	91.43	886	29.77	94.13	936	30.59	96.75	986	31.40	99.30
837	28.93	91.49	887	29.78	94.18	937	30.61	96.80	987	31.42	99.35
838	28.95	91.54	888	29.80	94.23	938	30.63	96.85	988	31.43	99.40
839	28.97	91.60	889	29.82	94.29	939	30.64	96.90	989	31.45	99.45
840	28.98	91.65	890	29.83	94.34	940	30.66	96.95	990	31.46	99.50
841	29.00	91.71	891	29.85	94.39	941	30.68	97.01	991	31.48	99.55
842	29.02	91.76	892	29.87	94.45	942	30.69	97.06	992	31.50	99.60
843	29.03	91.82	893	29.88	94.50	943	30.71	97.11	993	31.51	99.65
844	29.05	91.87	894	29.90	94.55	944	30.72	97.16	994	31.53	99.70
845	29.07	91.92	895	29.92	94.60	945	30.74	97.21	995	31.54	99.75
846	29.09	91.98	896	29.93	94.66	946	30.76	97.26	996	31.56	99.80
847	29.10	92.03	897	29.95	94.71	947	30.77	97.31	997	31.58	99.85
848	29.12	92.09	898	29.97	94.76	948	30.79	97.37	998	31.59	99.90
849	29.14	92.14	899	29.98	94.82	949	30.81	97.42	999	31.61	99.95
									1000	31.62	100.0